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On Integral Nonlocal Boundary Value Problems for some Partial Differential Equations

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ABSTRACT. The present paper deals with the results of investigation of nonclassical problems for elliptic and hyperbolic partial differential equations with integral nonlocal boundary conditions. Boundary value problems for elliptic equation on multidimensional cylindrical domain with one and two integral boundary conditions are considered. The nonclassical problems for elliptic equation are investigated applying variational approach in suitable Sobolev spaces and the existence and uniqueness results are proved. Nonclassical problems for multidimensional hyperbolic equation with integral boundary conditions are studied and the uniqueness of classical solution is proved. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *second-order elliptic and hyperbolic equations, integral boundary conditions, existence and uniqueness of solution.*

Boundary and initial-boundary value problems with nonclassical boundary conditions are often used to take into account some peculiarities of physical, chemical or other processes, which is impossible by applying classical boundary conditions. Nonclassical problem with integral boundary condition instead of classical Dirichlet or Neumann boundary condition for one-dimensional parabolic equation first was considered by J. Cannon [1] to describe the process of heat conduction, which was reduced to the integral equation and investigated in the spaces of classical functions. Integral nonlocal boundary conditions can be used when it is impossible to directly determine the values of the sought quantity on the boundary and we know its total amount or integral average on space domain, e.g., total energy, average temperature, total mass of impurities. Further, nonclassical problems with integral boundary conditions for various evolution equations were investigated in the papers [2-6]. Systematic investigation of nonlocal problems for elliptic equations was carried out by A. Bitsadze and A. Samarskii in the paper [7] applying methods of integral equations, where the nonclassical problem was studied for Laplace operator on rectangular domain with nonlocal condition connecting the values of the unknown function on one side of the rectangle with its values in the interior points of the rectangle and the classical Dirichlet boundary conditions given on the remaining three sides of the rectangle. The nonlocal problem considered in [7] was investigated by applying a different approach in the paper [8, 9], where iteration procedure was suggested which permits one to prove the existence of solution and also to construct an algorithm of solution of nonlocal problem. Later on, spatial nonlocal problem stated in [7] and its generalizations were studied for various equations of mathematical physics (see [10-17] and references given therein).

In the present paper we study integral nonlocal boundary value problems for multidimensional elliptic equation and nonclassical initial-boundary value problems with integral boundary conditions for hyperbolic equation. Note that papers, where integral nonlocal boundary value problems are considered, mainly deal with integral boundary conditions given on the whole space domain. We obtain variational formulation of integral nonlocal boundary value problem in corresponding weighted Sobolev spaces. We prove an embedding theorem, which permits one to show that the variational formulation of the nonclassical problem in weighted spaces is equivalent to the variational formulation of the nonlocal problem in corresponding spaces of vector-valued distributions with values in usual Sobolev spaces. We prove the uniqueness of solution applying formulation in weighted spaces and on the basis of suitable a priori estimates we obtain the existence of solution in corresponding spaces. We study nonclassical problems for hyperbolic equations with integral boundary conditions and prove the uniqueness result in the case of multidimenisonal space domain.

For bounded domain $\Omega \subset \mathbf{R}^p$, $p \geq 1$, with Lipschitz boundary $\Gamma = \partial\Omega$ we denote by $H^k(\Omega)$, $k \in \mathbf{N}$, the Sobolev space of order *k* and the closure of the set $D(\Omega)$ of infinitely differentiable functions with compact support in Ω in the space $H^k(\Omega)$ we denote by $H_0^k(\Omega)$. For any Banach space *X*, $C^0([0,\mu];X)$ denotes the space of continuous vector-functions on $[0,\mu]$ with values in *X*, $\|g(y)\|_X \in L^p(0,\mu)$ is the space of such vector-functions $g:(0,\mu) \to X$ that $\|g(y)\|_X \in L^p(0,\mu)$, $1 \leq p \leq \infty$. We denote by $g' = dg/dy$ the generalized derivative of $g \in L^p(0,\mu;X)$ in the sense of distributions $D'(0,\mu;X)$ on $(0,\mu)$ with values in *X*.

Let us consider a nonclassical problem for multidimensional elliptic equation

$$-b\frac{\partial^2 u}{\partial y^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j} \right) + a_0 u = f \quad \text{in} \quad \Omega = \omega \times (0,\mu), \tag{1}$$

with the following classical homogeneous Dirichlet and nonlocal integral boundary condition at one end of the cylinder $\Omega = \omega \times (0,\mu)$,

$$u = 0 \quad \text{on} \ \partial\omega \times (0,\mu) \cup \omega \times \{\mu\}, \qquad \int_0^{\xi_1} u(x,y)dy = 0, \quad x \in \omega, \tag{2}$$

where $\omega \subset \mathbf{R}^n$ $(n \geq 1)$ is a bounded domain with Lipschitz boundary $\partial\omega$, $0 < \xi_1 < \mu$ and $f:\Omega \to \mathbf{R}$, $b, a_{ij}, a_0 : \omega \to \mathbf{R}$ ($i,j = 1,\dots,n$) are given functions from suitable spaces. We investigate the existence and uniqueness of weak solution of nonlocal problem (1), (2) in corresponding spaces of distributions on the basis of variational formulation of the nonclassical problem. Assume that function *u* is twice continuously differentiable on $\bar{\Omega}$, functions a_{ij} ($i,j = 1,\dots,n$) are continuously differentiable on $\bar{\omega}$, functions *b*, a_0 are continuous on $\bar{\omega}$ and *f* is continuous on $\bar{\Omega}$. Let *v* be a continuously differentiable function on $\bar{\Omega}$, which satisfies conditions (2). Applying function *v*, let us define function $\bar{v}(x,y) = \int_y^{\xi_1} v(x,\alpha)d\alpha$ and function

$$\varphi_v(x,y) = \begin{cases} \dfrac{1}{\xi_1}\left(yv + \bar{v}\right), & 0 \leq y < \xi_1, \ x \in \omega, \\ v, & \xi_1 \leq y \leq \mu, \ x \in \omega. \end{cases}$$

Taking account of the nonlocal integral boundary condition (2), we shall have $\bar{v}(x,0) = 0$, for $x \in \omega$. Hence, $\varphi_v(x,y) = 0$ on $\partial\Omega$ and $\varphi_v(x,y)$ is a continuously differentiable function on $\bar{\Omega}$. Multiplying equation (1) by $\varphi_v(x,y)$, integrating on Ω and using Green's formula we obtain

$$\int_\omega \int_0^{\xi_1} \frac{y}{\xi_1}\left(b(x)\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + a_0(x)uv \right)dydx + \int_\omega \int_{\xi_1}^{\mu} \left(b(x)\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + a_0(x)uv \right)dydx +$$

$$+\int_\omega \int_0^{\xi_1} \frac{1}{\xi_1}\left(\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \bar{v}}{\partial x_i} + a_0(x)u\bar{v} \right)dydx = \int_\omega \int_0^{\xi_1} f(x,y)\frac{1}{\xi_1}\left(yv + \bar{v}\right)dydx + \int_\omega \int_{\xi_1}^{\mu} f(x,y)v(x,y)dydx. \tag{3}$$

Note that function $\varphi_v(x,y)$ can be equal to an arbitrary continuously differentiable function on $\overline{\Omega}$ vanishing in the neighbourhood of the boundary $\partial\Omega$. Indeed, for any given function $\varphi\in C_0^1(\overline{\Omega})$, we take

$$v(x,y)=\begin{cases}\varphi(x,\xi_1)-\int\limits_y^{\xi_1}\frac{\xi_1}{\alpha}\frac{\partial\varphi}{\partial y}(x,\alpha)d\alpha, & 0\le y<\xi_1,\ x\in\omega,\\ \varphi(x,y),\quad \xi_1\le y\le\mu,\ x\in\omega,\end{cases}$$

then $v(x,y)=0$ on $(x,y)\in\partial\omega\times(0,\mu)\cup\omega\times\{\mu\}$. Since $\varphi(x,\xi_1)-\int\limits_y^{\xi_1}\frac{\xi_1}{\alpha}\frac{\partial\varphi}{\partial y}(x,\alpha)d\alpha=\frac{\xi_1}{y}\varphi(x,y)-\int\limits_y^{\xi_1}\frac{\xi_1}{\alpha^2}\varphi(x,\alpha)d\alpha$, we have $v\in C^1(\overline{\Omega})$ and

$$\int\limits_0^{\xi_1}v(x,y)dy=\varphi(x,\xi_1)\xi_1-\int\limits_0^{\xi_1}\int\limits_y^{\xi_1}\frac{\xi_1}{\alpha}\frac{\partial\varphi}{\partial y}(x,\alpha)d\alpha dy=\varphi(x,\xi_1)\xi_1-\xi_1\int\limits_0^{\xi_1}\frac{\alpha}{\alpha}\frac{\partial\varphi}{\partial y}(x,\alpha)d\alpha=0.$$

Hence, function $v(x,y)$ satisfies conditions (2) and the corresponding function $\varphi_v(x,y)$ equals the given function $\varphi(x,y)$. Consequently, from equation (3), applying Green's formula and taking account of the density of $C_0^1(\overline{\Omega})$ in $L^2(\Omega)$, we deduce that sufficiently smooth solution $u(x,y)$ of (3) satisfies equation (1).

To define weak solution of the nonclassical problem (1), (2) on the basis of equation (3) for abstract Hilbert space X let us introduce the following space of vector-valued distributions

$$V_1(0,\mu;X)=\{v\in D'(0,\mu;X)\mid d^\alpha v/dy^\alpha\sqrt{y}\in L^2(0,\mu;X),\alpha=0,1\},$$

which is a Hilbert space equipped with the norm

$$\|v\|_{V_1(0,\mu;X)}=(\|v\sqrt{y}\|^2_{L^2(0,\mu;X)}+\|dv/dy\sqrt{y}\|^2_{L^2(0,\mu;X)})^{1/2}.$$

Note that the functions from the space $V_1(0,\mu;X)$ may be discontinuous for $y=0$, however the following embedding theorem is valid.

Theorem 1. *Functions* $v\in V_1(0,\mu;X)$ *belong to the space* $L^2(0,\mu;X)$ *and the embedding* $V_1(0,\mu;X)\to L^2(0,\mu;X)$ *is continuous.*

From equation (3) it follows that in the space of sufficiently smooth functions the integral nonlocal problem (1), (2) is equivalent to the following problem in weighted Sobolev spaces: Find function $u\in\tilde{V}_1=\{v\in V_1(0,\mu;H_0^1(\omega))\mid v(\mu)=0$ in $H_0^1(\omega)$, $\partial u/\partial y\in V_1(0,\mu;L^2(\omega))$, $\int\limits_0^{\xi_1}u(y)dy=0$ in $H_0^1(\omega)\}$, which satisfies equation (3), for all $v\in\tilde{V}_1$, where each function from the space $L^2(\Omega)$ we identify with function from $L^2(0,\mu;H_0^1(\omega))$. Since $u\in L^2(\mu/2,\mu;H_0^1(\omega))$, $\partial u/\partial y\in L^2(\mu/2,\mu;H_0^1(\omega))$, from the embedding theorem [18] it follows that $u\in C^0([\mu/2,\mu];H_0^1(\omega))$ and applying Theorem 1 we deduce that the definition of the space $\tilde{V}_1$ is correct.

Note that, if $v\in\tilde{V}_1$, then by Theorem 1 $v\in L^2(0,\mu;H_0^1(\omega))$, $\partial v/\partial y\in L^2(0,\mu;L^2(\omega))$ and applying the embedding theorem we obtain that $v\in C^0([0,\mu];L^2(\omega))$. Consequently, function $\overline{v}(y)=\int\limits_y^{\xi_1}v(\alpha)d\alpha\in C^0([0,\mu];H_0^1(\omega))$, $\partial\overline{v}/\partial y=-v\in L^2(0,\xi_1;H_0^1(\omega))$, $\overline{v}(0)=\overline{v}(\xi_1)=0$ in $H_0^1(\omega)$, and the function φ_v belongs to $L^2(0,\mu;H_0^1(\omega))\cap C^0([0,\mu];L^2(\omega))$, $\varphi_v(x,0)=\varphi_v(x,\mu)=0$ almost everywhere in ω and $\partial\varphi_v/\partial y\in L^2(0,\mu;L^2(\omega))$. Hence, the function φ_v corresponding to $v\in\tilde{V}_1$ belongs to $H_0^1(\Omega)$. Since the set $D(\Omega)\subset C_0^1(\overline{\Omega})$ of infinitely differentiable functions with compact support in Ω is dense in $H_0^1(\Omega)$, the above mentioned formulation of nonlocal problem in weighted spaces is equivalent to the following variational formulation in usual Sobolev spaces: Find function $u \in L^2(0,\mu;H_0^1(\omega))$, $\partial u/\partial y \in L^2(0,\mu;L^2(\omega))$, which satisfies the equation

$$\int_0^\mu\int_\omega\left(b\frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial y}+\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}+a_0 u\varphi\right)dxdy=\int_0^\mu\int_\omega f(x,y)\varphi(x,y)dxdy,\quad \forall\varphi\in H_0^1(\Omega), \tag{4}$$

nonlocal boundary conditions (2) and possesses additional regularity properties $\partial u/\partial y\sqrt{y}\in L^2(0,\mu;H_0^1(\omega))$, $\partial^2 u/\partial y^2\sqrt{y}\in L^2(0,\mu;L^2(\omega))$.

We assume that coefficients $b, a_{ij}, a_0 \in L^\infty(\omega)$ and satisfy the following symmetry and positive definiteness conditions

$$b(x)\ge c_b,\ \ a_0(x)\ge 0,\ \ a_{ij}(x)=a_{ji}(x),\ \ \sum_{i,j=1}^n a_{ij}(x)\xi_j\xi_i\ge c_a\sum_{i=1}^n|\xi_i|^2,\ \ \forall\xi_i\in\mathbf{R}, i,j=\overline{1,n}, \tag{5}$$

for almost all $x\in\omega$, $c_a = const > 0$, $c_b = const > 0$. Letting $v = u$ in the equation (3) we obtain

$$\int_\omega\int_0^{\xi_1}\frac{y}{\xi_1}\left(b(x)\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}+\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}+a_0(x)uu\right)dydx+\int_\omega\int_{\xi_1}^\mu\left(b(x)\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}+\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}+a_0(x)uu\right)dydx+$$

$$+\int_\omega\int_0^{\xi_1}\frac{1}{\xi_1}\left(\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \overline{u}}{\partial x_i}+a_0(x)u\overline{u}\right)dydx=\int_\omega\int_0^{\xi_1}f(x,y)\frac{1}{\xi_1}\left(yu+\overline{u}\right)dydx+\int_\omega\int_{\xi_1}^\mu f(x,y)u(x,y)dydx, \tag{6}$$

where $\overline{u}(x,y)=\int_y^{\xi_1}u(x,\alpha)d\alpha$ and $\overline{u}\in C^0([0,\xi_1];H_0^1(\omega))$, $\partial\overline{u}/\partial y=-u\in L^2(0,\xi_1;H_0^1(\omega))$, $\overline{u}(0)=\overline{u}(\xi_1)=0$ in $H^1(\omega)$.

Therefore, by using the formula for integration by parts, for the third integral in the left hand part of the equation (6) we have

$$\int_\omega\int_0^{\xi_1}\left(\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \overline{u}}{\partial x_i}+a_0u\overline{u}\right)dydx=-\int_\omega\int_0^{\xi_1}\left(\sum_{i,j=1}^n a_{ij}\frac{\partial^2\overline{u}}{\partial y\partial x_j}\frac{\partial \overline{u}}{\partial x_i}+a_0\frac{\partial\overline{u}}{\partial y}\overline{u}\right)dydx=$$

$$=-\int_\omega\left[\sum_{i,j=1}^n\frac{a_{ij}}{2}\left(\frac{\partial\overline{u}}{\partial x_j}(x,\xi_1)\frac{\partial\overline{u}}{\partial x_i}(x,\xi_1)-\frac{\partial\overline{u}}{\partial x_j}(x,0)\frac{\partial\overline{u}}{\partial x_i}(x,0)\right)+\frac{a_0}{2}(|\overline{u}(x,\xi_1)|^2-|\overline{u}(x,0)|^2)\right]dx=0.$$

Consequently, if $f\equiv 0$, then from conditions (5) it follows that $u\equiv 0$, i.e. the nonlocal problem has at most one solution. The existence of solution can be proved applying suitable a priori estimates corresponding to the integral boundary condition. More precisely, the following theorem is valid.

Theorem 2. *If $f\in L^2(\Omega)$ and the coefficients $b, a_{ij}, a_0\in L^\infty(\omega)$ ($i,j=1,...,n$) satisfy the conditions (5), then the nonlocal problem (2), (4) has a unique solution.*

Applying the approach similar to the one used for investigation of the nonclassical problem (1), (2) with one integral boundary condition we study nonclassical problem for elliptic equation (1) with the following nonlocal boundary conditions involving two integral boundary conditions

$$u=0 \quad\text{on } \partial\omega\times(0,\mu),\qquad \int_0^{\xi_1}u(x,y)dy=0,\quad \int_{\xi_2}^\mu u(x,y)dy=0,\quad x\in\omega, \tag{7}$$

where $0<\xi_1<\xi_2<\mu$. In order to obtain variational formulation of the nonclassical problem (1), (7) let us assume that u is a twice continuously differentiable function on $\bar{\Omega}$ and v is a continuously differentiable function on $\bar{\Omega}$, which satisfies conditions (7). Let us define the functions $\bar{v}(x,y)=\int\limits_y^{\xi_1} v(x,\alpha)d\alpha$, $\tilde{v}(x,y)=\int\limits_{\xi_2}^{y} v(x,\alpha)d\alpha$ and applying them we construct the following function

$$\psi_v(x,y)=\begin{cases}\dfrac{1}{\xi_1}\left(yv+\bar{v}\right), & 0\le y<\xi_1,\ x\in\omega,\\ v, & \xi_1\le y\le\xi_2,\ x\in\omega,\\ \dfrac{1}{\mu-\xi_2}\left((\mu-y)v+\tilde{v}\right), & \xi_2<y\le\mu,\ x\in\omega.\end{cases}$$

From nonlocal boundary conditions (7) it follows that $\psi_v(x,y)=0$ on $\partial\Omega$ and $\psi_v(x,y)$ is a continuously differentiable function on $\bar{\Omega}$. Multiplying equation (1) by $\psi_v(x,y)$, integrating on Ω and using Green's formula, we obtain

$$\int\limits_\omega\int\limits_0^{\xi_1}\frac{y}{\xi_1}\left(b\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}+a_0uv\right)dydx+\int\limits_\omega\int\limits_{\xi_1}^{\xi_2}\left(b\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}+a_0uv\right)dydx+$$

$$+\int\limits_\omega\int\limits_{\xi_2}^{\mu}\frac{\mu-y}{\mu-\xi_2}\left(b\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}+a_0uv\right)dydx+\int\limits_\omega\int\limits_0^{\xi_1}\frac{1}{\xi_1}\left(\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \bar{v}}{\partial x_i}+a_0u\bar{v}\right)dydx+$$

$$+\int\limits_\omega\int\limits_{\xi_2}^{\mu}\frac{1}{\mu-\xi_2}\left(\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \tilde{v}}{\partial x_i}+a_0u\tilde{v}\right)dydx=\int\limits_\omega\int\limits_0^{\mu}f(x,y)\psi_v(x,y)dydx. \qquad (8)$$

Note that function $\psi_v(x,y)$ can be equal to an arbitrary continuously differentiable function on $\bar{\Omega}$ vanishing in the neighbourhood of the boundary $\partial\Omega$ and since $C_0^1(\bar{\Omega})$ is dense in $L^2(\Omega)$ we obtain that (8) is equivalent to equation (1) in the space of sufficiently smooth functions.

In order to investigate nonlocal problem (1), (7) in corresponding spaces of distributions for abstract Hilbert space X let us define the following weighted space of vector-valued distributions

$$V_2(0,\mu;X)=\{v\in D'(0,\mu;X)\,|\,d^\alpha v/dy^\alpha\sqrt{y}\in L^2(0,\frac{3\mu}{4};X),d^\alpha v/dy^\alpha\sqrt{\mu-y}\in L^2(\frac{\mu}{4},\mu;X),\alpha=0,1\},$$

which is a Hilbert space equipped with the norm

$$\|v\|_{V_2(0,\mu;X)}=(\|v\sqrt{y}\|^2_{L^2(0,\frac{3\mu}{4};X)}+\|dv/dy\sqrt{y}\|^2_{L^2(0,\frac{3\mu}{4};X)}+\|v\sqrt{\mu-y}\|^2_{L^2(\frac{\mu}{4},\mu;X)}+\|dv/dy\sqrt{\mu-y}\|^2_{L^2(\frac{\mu}{4},\mu;X)})^{1/2}.$$

For weighted space $V_2(0,\mu;X)$ the following embedding theorem is valid.

Theorem 3. *Functions $v\in V_2(0,\mu;X)$ belong to the space $L^2(0,\mu;X)$ and the embedding $V_2(0,\mu;X)\to L^2(0,\mu;X)$ is continuous.*

Nonclassical problem for elliptic equation (1) with integral boundary conditions (7) admits the following variational formulation in weighted Sobolev spaces: Find function $u\in\tilde{V}_2=\{v\in V_2(0,\mu;H_0^1(\omega))\,|\,\partial u/\partial y\in V_2(0,\mu;L^2(\omega)),$ $\int\limits_0^{\xi_1}u(y)dy=0$ and $\int\limits_{\xi_2}^{\mu}u(y)dy=0$ in $H_0^1(\omega)\}$, which satisfies equation (8), for all $v\in\tilde{V}_2$.

For any function $v\in\tilde{V}_2$, by Theorem 3, $v\in L^2(0,\mu;H_0^1(\omega))$, $\partial v/\partial y\in L^2(0,\mu;L^2(\omega))$ and embedding theorem [18] implies that $v \in C^0([0,\mu];L^2(\omega))$. Consequently, functions $\bar{v}(y) \in C^0([0,\xi_1];H_0^1(\omega))$, $\partial \bar{v}/\partial y = -v \in L^2(0,\xi_1;H_0^1(\omega))$, $\tilde{v}(y) \in C^0([\xi_2,\mu];H_0^1(\omega))$, $\partial \tilde{v}/\partial y = v \in L^2(\xi_2,\mu;H_0^1(\omega))$, $\bar{v}(0) = \bar{v}(\xi_1) = \tilde{v}(\xi_2) = \tilde{v}(\mu) = 0$ in $H_0^1(\omega)$, and the function ψ_v belongs to $L^2(0,\mu;H_0^1(\omega)) \cap C^0([0,\mu];L^2(\omega))$, $\psi_v(x,0) = \psi_v(x,\mu) = 0$ almost everywhere in ω and $\partial\psi_v/\partial y \in L^2(0,\mu;L^2(\omega))$. Hence, variational formulation of an integral nonlocal problem in weighted spaces is equivalent to the following variational formulation in usual Sobolev spaces: Find function $u \in L^2(0,\mu;H_0^1(\omega))$, $\partial u/\partial y \in L^2(0,\mu;L^2(\omega))$, which satisfies the equation

$$\int_0^\mu\int_\omega \left(b\frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial y} + \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + a_0 u\varphi \right) dxdy = \int_0^\mu\int_\omega f(x,y)\varphi(x,y)dxdy, \quad \forall \varphi \in H_0^1(\Omega), \tag{9}$$

nonlocal boundary conditions (7) and possesses additional regularity properties $\partial u/\partial y\sqrt{y} \in L^2(0,\frac{3\mu}{4};H_0^1(\omega))$, $\partial^2 u/\partial y^2\sqrt{y} \in L^2(0,\frac{3\mu}{4};L^2(\omega))$, $\partial u/\partial y\sqrt{\mu-y} \in L^2(\frac{\mu}{4},\mu;H_0^1(\omega))$, $\partial^2 u/\partial y^2\sqrt{\mu-y} \in L^2(\frac{\mu}{4},\mu;L^2(\omega))$.

The variational formulation (8) with $v = u$ permits one to obtain the uniqueness result and on the basis of suitable a priori estimates we prove the following existence and uniqueness theorem.

Theorem 4. *If $f \in L^2(\Omega)$ and the coefficients $b, a_{ij}, a_0 \in L^\infty(\omega)$ $(i,j=1,...,n)$ satisfy the conditions (5), then the nonlocal problem (7), (9) has a unique solution.*

Let us now consider a nonclassical problem for multidimensional hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - b\frac{\partial^2 u}{\partial y^2} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j} \right) + a_0 u = f \quad \text{in} \quad \Omega\times(0,T), \tag{10}$$

with integral nonlocal boundary conditions and classical initial conditions

$$u(x,y,t) = 0 \quad \text{on} \quad (\partial\omega\times(0,\mu)\cup\omega\times\{\mu\})\times(0,T), \quad \int_0^{\xi_1} u(x,y,t)dy = 0, \quad \text{in} \quad \omega\times(0,T), \tag{11}$$

$$u(x,y,0) = u_0(x,y), \quad \frac{\partial u}{\partial t}(x,y,0) = u_1(x,y), \qquad (x,y)\in\Omega, \tag{12}$$

where $0 < \xi_1 < \mu$ and $f:\Omega\times(0,T)\to\mathbf{R}$, $b, a_{ij}, a_0 : \omega\times(0,T)\to\mathbf{R}$ $(i,j=1,...,n)$ are given continuous functions and u, $\partial u/\partial y : \bar{\Omega}\times[0,T]\to\mathbf{R}$ are twice continuously differentiable functions and u is a classical solution of the nonclassical problem (10)-(12). The following uniqueness theorem is valid for a nonlocal initial-boundary value problem.

Theorem 5. *If the coefficients b, a_{ij}, a_0 $(i,j=1,...,n)$ are continuously differentiable with respect to the time variable t and conditions (5) are fulfilled, then the nonclassical problem for hyperbolic equation (10) with integral nonlocal boundary conditions (11) and initial conditions (12) has at most one solution.*

Note that the result on the uniqueness of solution of an integral nonlocal initial-boundary value problem for hyperbolic equation (10) is also valid for a nonclassical problem with two integral nonlocal boundary conditions of the form (7).

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ზოგიერთი კერძოწარმოებულიანი დიფერენციალური განტოლებისათვის ინტეგრალური არალოკალური სასაზღვრო ამოცანების შესახებ

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ნაშრომში წარმოდგენილია ელიფსური და ჰიპერბოლური კერძოწარმოებულიანი დიფერენციალური განტოლებებისათვის ინტეგრალური არალოკალური სასაზღვრო პირობებით არაკლასიკური ამოცანების გამოკვლევის შედეგები. განხილულია სასაზღვრო ამოცანები ელიფსური განტოლებისათვის მრავალგანზომილებიან ცილინდრულ არეში ერთი და ორი ინტეგრალური სასაზღვრო პირობებით. არაკლასიკური ამოცანები ელიფსური განტოლებისათვის გამოკვლეულია ვარიაციული მიდგომის გამოყენებით სათანადო სობოლევის სივრცეებში და დამტკიცებულია ამონახსნის არსებობის და ერთადერთობის შედეგები. შესწავლილია არაკლასიკური ამოცანები მრავალგანზომილებიანი ჰიპერბოლური განტოლებისათვის ინტეგრალური სასაზღვრო პირობებით და დამტკიცებულია კლასიკური ამონახსნის ერთადერთობა.

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