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Mathematics

Splitting Fields for Crossed Group Rings

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ABSTRACT. It is proved that in certain cases a crossed group ring over a field of a positive characteristic has a purely inseparable splitting field. For such crossed group ring some properties of its radical are investigated.

Key words: *crossed group ring, splitting field, purely inseparable extension.*

Let F be a field, G – a finite group, $\sigma: G \to Aut(F)$– a group homomorphism and $\rho: G\times G \to U(F)$ - a 2-cocycle, i.e.

$$\rho \in Z^2(G,U(F)),\quad \rho(x,yz)=\rho(x,y)\rho(xy,z)\rho(y,z)^{\sigma(x^{-1})}$$

(further we shall write α^x instead of $\alpha^{\sigma(x)}$, where $\alpha \in F$, $x \in G$).

Let $F[G,\sigma,\rho]$ denote a crossed group ring with a multiplication

$$\alpha_1\overline{x_1}\cdot\alpha_2\overline{x_2}=\alpha_1\alpha_2^{x_1}\rho(x_1,x_2)\overline{x_1x_2},$$

where $\alpha_i \in F$, $x_i \in G$, $\overline{x_i} \in F[G,\sigma,\rho]$. It is well known ([1], [2]) that if $Ker\,\sigma=1$ then $F[G,\sigma,\rho]$ is a central simple algebra and F is a splitting field for $F[G,\sigma,\rho]$; this means that if we denote $K=F^G$ when in an algebra

$$F[G,\sigma,\rho]\otimes_K F \cong (F\otimes_K F)[G\times 1,\sigma\times 1,\rho\times 1]$$

which is really a central simple algebra we have $\rho\times 1 \sim 1$, i.e.

$$(F\otimes_K F)[G\times 1,\sigma\times 1,\rho\times 1]\cong (F\otimes_K F)[G\times 1,\sigma\times 1].$$

In the other particular case when $Ker\,\sigma=G$, i.e. when G acts on F trivially and $charF=p>0$, from $(G:1)=p^n$ it follows [3] that there exists a purely inseparable extension $E\supseteq F$ such that E splits $F[G,\rho]$, i.e.

$$E\otimes_F F[G,\rho]\cong E[G,\rho]\cong E[G].$$

The next theorem joins these two extreme cases together for fields of a positive characteristic:

Theorem 1. *Let $charF=p>0$, $Ker\,\sigma=H$, $(G:H)=m$, $(H:1)=p^n$. Then there exists a purely inseparable extension of fields $E\supseteq F$ such that*

a. It is possible to lift $F[G,\sigma,\rho]$ up to $E[G,\sigma,\rho]$;

b. E splits $E[G,\sigma,\rho]$ i.e. if $K=E^G$, then

$$E[G,\sigma,\rho]\otimes_K E\cong(E\otimes_K E)[G\times 1,\sigma\times 1,\rho\times 1]\cong(E\otimes_K E)[G\times 1,\sigma\times 1].$$

Proof. Let $G=\{1=g_1,g_2,...,g_{mp^n}\}$. Let E be a splitting field of the equations

$$z^{p^n}-g_{i_1}...g_{i_t}\prod_{h\in H}\rho(hx,y)=0, \tag{1}$$

where g_{i_j}, $x, y \in G$. Then $E\supset F$ is a purely inseparable extension of the fields ([4], §7.7) and therefore for any $g\in G$ there exists an extension of the automorphism $\sigma(g):F\to F$ up to the automorphism $\sigma(g):E\to E$; really, if $\overline{E}=\overline{F}$ is an algebraic closure of F, then there exists only one automorphism $\sigma(g):E\to\overline{F}$ which is an extension of $\sigma(g):F\to F\to\overline{F}$; therefore $\sigma(E)=E$ because $E\supset F$ is simultaneously a normal extension of fields. Consequently, we can construct a crossed group ring $E[G, \sigma, \rho]$, which is an extension of $F[G, \sigma, \rho]$.

Now let us consider

$$E[G,\sigma,\rho]\otimes_K E\cong\left(E\otimes_K E\right)[G\times 1,\sigma\times 1,\rho\times 1],$$

where $K=E^G$. We must prove that

$$\rho\times 1\cong 1\times 1.$$

Because E is a Galois extension of K relative to G/H, then ([2], Lemma A.10) there exist elements

$$e_{\bar{x}}\in E\otimes_K E,\quad \bar{x}\in G/H,$$

such that

(i) $e_{\bar{x}}e_{\bar{y}}=0$ if $\bar{x}\neq\bar{y}$, and $e_{\bar{x}}^2=e_{\bar{x}}$, $\sum_{x\in G/H}e_{\bar{x}}=1\otimes 1$;

(ii) $(\alpha^{\bar{x}}\otimes 1)e_{\bar{x}}=(1\otimes\alpha)e_{\bar{x}}$, $\alpha\in E$;

(iii) $e_{\bar{z}}^{\bar{x}\times\bar{y}}=e_{\overline{xzy^{-1}}}$, $x,y,z\in G$.

Let $G=x_1H\dot{\cup}x_2H\dot{\cup}\cdots\dot{\cup}x_mH$ and let us define $\gamma:G\times 1\to U(E\otimes_K E)$ as

$$\lambda(y\times 1)=\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_jx_i^{-1},y)}\right)e_{\bar{x}_i}. \tag{2}$$

From (1) it follows that the root is well defined in (2). From (i) it follows that $\gamma(y\times 1)\in U(E\otimes_K E)$. Indeed if $0\neq\alpha_i\in E$, then from (i) it follows that

$$\sum_{i=1}^{m}(1\otimes_K\alpha_i)e_{\bar{x}_i}\cdot\sum_{i=1}^{m}(1\otimes_K\alpha_i^{-1})e_{\bar{x}_i}=\sum_{i=1}^{m}(1\otimes 1)e_{\bar{x}_i}=1\otimes 1,$$

i. e.

$$\left(\sum_{i=1}^{m}(1\otimes_K\alpha_i)e_{\bar{x}_i}\right)^{-1}=\sum_{i=1}^{m}(1\otimes_K\alpha_i^{-1})e_{\bar{x}_i}. \tag{3}$$

Let us prove

$$\partial(\gamma)(y\times 1,z\times 1)=\gamma(y\times 1)\gamma(z\times 1)^{y\times 1}\gamma(yz\times 1)^{-1}=\rho(x,y)\times 1,$$

i. e. $\rho\times 1\cong 1\times 1$.

Indeed, from (iii) it follows $e_{\bar{x}_i}^{\bar{y}\times 1}=e_{\overline{x_i y}}$; therefore

$$\partial(\gamma)(y\times 1,z\times 1)=\gamma(y\times 1)\gamma(z\times 1)^{y\times 1}\gamma(yz\times 1)^{-1}=\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},y)}\right)e_{\bar{x}_i}\right\}\times$$

$$\times\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},z)}\right)e_{\bar{x}_i}\right\}^{y\times 1}\times\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},yz)}\right)e_{\bar{x}_i}\right\}^{-1}=$$

$$=\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},y)}\right)e_{\bar{x}_i}\right\}\times\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},z)}\right)e_{\overline{yx_i}}\right\}\times$$

$$\times\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},yz)}\right)e_{\bar{x}_i}\right\}^{-1}.$$

Let us denote $\bar{y}\bar{x}_i=\overline{yx_i}=\bar{x}_{r_i}$; then $\bar{x}_i=\bar{y}^{-1}\bar{x}_{r_i}=\overline{y^{-1}x_{r_i}}$, $x_i=y^{-1}x_{r_i}h_{s_i}$, $h_{s_i}\in H$ (here $\bar{y},\ldots\in G/H$); then it is obvious that the last expression is equal to

$$\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},y)}\right)e_{\bar{x}_i}\right\}\times\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j h_{s_i}^{-1}x_{r_i}^{-1}y,z)}\right)e_{\bar{x}_{r_i}}\right\}\times\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},yz)}\right)e_{\bar{x}_i}\right\}^{-1}=$$

$$=\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},y)}\right)e_{\bar{x}_i}\right\}\times\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1}y,z)}\right)e_{\bar{x}_i}\right\}\times\left\{\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},yz)}\right)e_{\bar{x}_i}\right\}^{-1}.$$

From (3) and (i) it follows

$$\partial(\gamma)(y\times 1,z\times 1)=\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(h_j x_i^{-1},y)\rho(h_j x_i^{-1}y,z)\rho(h_j x_i^{-1},yz)^{-1}}\right)e_{\bar{x}_i}.$$

Because ρ is a 2-cocycle and H acts trivially on K and consequently on E, from (ii) and (i) it follows that

$$\partial(\gamma)(y\times 1,z\times 1)=\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(y,z)^{h_j x_i^{-1}}}\right)e_{\bar{x}_i}=\sum_{i=1}^{m}\left(1\otimes\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(y,z)^{x_i^{-1}}}\right)e_{\bar{x}_i}=$$

$$=\sum_{i=1}^{m}\left(\left(\sqrt[p^n]{\prod_{j=1}^{p^n}\rho(y,z)^{x_i^{-1}}}\right)^{x_i}\otimes 1\right)e_{\bar{x}_i}=\sum_{i=1}^{m}(\rho(y,z)\otimes 1)e_{\bar{x}_i}=\rho(y,z)\otimes 1.$$

Theorem 1 is proved.

Theorem 2. *Let E be a field, char E*$=p>0$. Let *G be a finite group,* $\sigma: G\to AutF$ *– a group homomorphism and* $Ker\sigma=H$ – a p-group. *Then* $rad(E[G,\sigma])=E[G,\sigma]\sum_{x\in H}(x-1)$.

Proof. Let us consider an essential morphism

$$\pi : E[G,\sigma] \to E[G/H, \bar{\sigma}], \tag{4}$$

where $\alpha^{\bar{\sigma}(\bar{x})} = \alpha^{\sigma(x)}$. It is clear that

$$Ker\pi = E[G,\sigma] \sum_{x\in H\setminus\{1\}} (x-1).$$

Really if $G = y_1 H \dot{\cup} y_2 H \dot{\cup} \cdots \dot{\cup} y_m H$ and

$$z = \sum_{y\in G} \alpha_y y = \sum_{i=1}^{m} \sum_{x\in H} \alpha_{y_i x} y_i x \in E[G,\sigma],$$

then $\pi(z) = \sum_{i=1}^{m} \sum_{x\in H} \alpha_{y_i x} \pi(y_i)$, and from $\pi(z) = 0$ it follows that $\alpha_{y_i} = -\sum_{x\in H\setminus\{1\}} \alpha_{y_i x}$; consequently

$$z = \sum_{i=1}^{m} \sum_{x\in H\setminus\{1\}} (\alpha_{y_i x} y_i x - \alpha_{y_i x} y_i) = \sum_{i=1}^{m} \sum_{x\in H\setminus\{1\}} \alpha_{y_i x} y_i (x-1) \in E[G,\sigma] \sum_{x\in H\setminus\{1\}} (x-1).$$

Because $E[G/H, \bar{\sigma}]$ is a simple (and central) algebra, from (4) it follows that

$$Ker(\pi) \supseteq rad(E[G,\sigma]).$$

Now let us prove $Ker(\pi) \subseteq rad(E[G,\sigma])$. Let us denote $B = \sum_{x\in H\setminus\{1\}} E(x-1)$; it is clear that *B is* a finite dimensional associative algebra without unit because $H = Ker(\sigma)$ and $(x-1)(y-1) = (xy-1)-(x-1)-(y-1)$. If $(H:1) = p^l$ then $(x-1)^{p^l} = x^{p^l} - 1$. Therefore *B* has a basis all elements of which are nilpotent. Then from Wedderburn's theorem [5, Theorem 27.27] it follows that there exists $k \in \boldsymbol{N}$ such that $B^k = 0$. It is clear that $Ker(\pi) = E[G,\sigma]\cdot B = B \cdot E[G,\sigma]$ from which it follows that

$$(Ker(\pi))^k = (E[G,\sigma]\cdot B)^k = E[G,\sigma]^k \cdot B^k = 0$$

and consequently $Ker(\pi) \subseteq rad(E[G,\sigma])$, $Ker(\pi) = rad(E[G,\sigma])$.

Corollary. $rad((E \otimes_K E)[G\times 1, \sigma\times 1]) = E[G\times 1, \sigma \times 1] \sum_{x\in H} (x-1)$.

Proof. If *A* is an algebra over a field *K* and $E \supset K$ is a field extension, then $rad(E\otimes_K A) \cong E \otimes_K rad(A)$, and Corollary follows from the fact that $E[G,\sigma]\otimes_K E \cong (E\otimes_K E)[G\times 1, \sigma\times 1]$.

Theorem 3. *In the conditions of Theorem 1 algebras* $E[G,\sigma,\rho]/rad(E[G,\sigma,\rho])$ *and* $F[G,\sigma,\rho]/rad(F[G,\sigma,\rho])$ *are central simple algebras.*

Proof. If *E* is the field from Theorem 1, then from the existence of an algebra map

$$E[G,\sigma,\rho] \to E \otimes_K E[G,\sigma,\rho]$$

It follows that

$$rad(E[G,\sigma,\rho]) = E[G,\sigma,\rho] \cap rad(E\otimes_K E[G,\sigma,\rho]).$$

Consequently we can construct an exact sequence of algebras

$$0 \to E[G,\sigma,\rho]/rad\,(E[G,\sigma,\rho]) \to (E\otimes_K E[G,\sigma,\rho])/rad((E\otimes_K E[G,\sigma,\rho])). \tag{5}$$

But *E* is a splitting field for $E[G,\sigma,\rho]$, therefore $E\otimes_K E[G,\sigma,\rho] \cong E\otimes_K E[G,\sigma]$ and from (5) it follows that there exists an exact sequence

$$0 \to E[G,\sigma,\rho]/rad\,(E[G,\sigma,\rho]) \to (E \otimes_K E[G,\sigma])/rad((E \otimes_K E[G,\sigma]))\,.$$

From Theorem 2 it follows that

$$(E \otimes_K E[G,\sigma])/rad((E \otimes_K E[G,\sigma])) \cong E \otimes_K E[G/H,\overline{\sigma}]$$

and we have an exact sequence of the algebras

$$0 \to E[G,\sigma,\rho]/rad\,(E[G,\sigma,\rho]) \to E \otimes_K E[G/H,\overline{\sigma}]\,.$$

The algebra $E[G,\sigma,\rho]/rad(E[G,\sigma,\rho])$, which is semi-simple, actually must be a simple algebra because

$$E \otimes_K E[G/H,\overline{\sigma}] \cong E \otimes_K E[G,\sigma,\rho]/rad(E[G,\sigma,\rho])$$

is a central simple algebra, and the theorem is proved for $E[G,\sigma,\rho]/rad(E[G,\sigma,\rho])$. The proof for $F[G,\sigma,\rho]/rad(F[G,\sigma,\rho])$ is similar – we must consider a map

$$0 \to F[G,\sigma,\rho]/rad(F[G,\sigma,\rho]) \to E[G,\sigma,\rho]/rad(E[G,\sigma,\rho])\,. \quad (6)$$

Then the left hand in (6) must be a central simple algebra because the right hand is a central simple algebra.

მათემატიკა

ჯვარედინი ჯგუფური რგოლების გახლეჩის ველები

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(წარმოდგენილია აკადემიკოს ხ. ინასარიძის მიერ)

ნაშრომში დამტკიცებულია, რომ გარკვეულ პირობებში დადებით მახასიათებლიან ველზე განსაზღვრულ ჯვარედინ ჯგუფურ რგოლს აქვს სუფთად არასეპარაბელური გახლეჩის ველი. ეს შედეგი გამოყენებულია იმის დასამტკიცებლად, რომ არსებობს გარკვეული მსგავსება ასეთი ჯვარედინი ჯგუფური რგოლის (იგი ლოკალური რგოლია) რადიკალსა და სასრული p-ჯგუფის ჯგუფური რგოლის რადიკალს შორის, როცა ჯგუფური რგოლი აიღება კოეფიციენტებით დადებით მახასიათებლიან ველში.

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