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Mathematics

Generalized Theta-Functions with Characteristics and Cusp Forms Corresponding to Quadratic Forms in Nine Variables

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ABSTRACT. The modular properties of generalized theta-functions with characteristics are used to build a cusp form of weight $\frac{9}{2}$ on the congruence subgroup $r_0(48)$. It gives the opportunity of obtaining formulas for the number of representations of positive integers by quadratic forms in nine variables. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *cusp form, modular form, positive definite quadratic form, generalized theta-function with characteristics.*

1. Introduction

Let $A=(a_{jk})$ be an integral square symmetric matrix of *s* rows and *s* columns the diagonal elements of which are even integers. If *x* is a column variable vector with *s* components then the matrix product $\frac{1}{2}x'Ax=f(x)$ is a quadratic form. We shall suppose this quadratic form to be definite and positive. The determinant Δ of *A* is then a positive integer.

An integral column vector *a* with *s* components will be called special (with respect to *A*) if *Aa* is divisible by *N*, where *N* is a level of *f*(*x*) (it is a least integer for which NA^{-1} is an integral matrix with even diagonal elements).

Now let τ be a complex variable with a positive imaginary part. Let *P*(*x*) be a spherical function of order ν with respect to *A*. Let *g* and *h* be special column vectors with respect to *A*. Then we define the generalized theta-function with characteristics ([1], p.438).

$$\vartheta_{gh}\left(\tau;P_\nu,f(x)\right)=\sum_{x\equiv g(\mathrm{mod}\,N)}(-1)^{\frac{h'A(x-g)}{N^2}}P_\nu(x)e^{\frac{\pi\, i\tau\, x'Ax}{N^2}}. \tag{1}$$

It is well known that $r(n;f)$, the number of representations of a positive integer *n* by a positive definite quadratic form *f*, can be expressed as the sum

$$r(n;\ f)=\rho(n;\ f)+\nu(n;\ f),$$

where $\rho(n;\ f)$ is a singular series. This series has been exhaustively studied and formulas for computing it are known ([2,3]). The second summand $\nu(n;\ f)$ is a Fourier coefficient of a cusp form. It can be expressed in terms of modular forms as follows:

$$\vartheta(\tau;\ f)=E(\tau;\ f)+X(\tau),$$

$$\vartheta(\tau;\ f)=1+\sum_{n=1}^{\infty}r(n;\ f)Q^{n}, \tag{2}$$

where $Q=e^{2\pi i\tau}$, $X(\tau)$ is a cusp form and

$$E(\tau;\ f)=1+\sum_{n=1}^{\infty}\rho(n;\ f)Q^{n}$$

is the Eisenstein series corresponding to *f*. If the genus of the quadratic form contains one class, then, according to Siegel's theorem ([4-6]), $\vartheta(\tau;\ f)=E(\tau;\ f)$ and therefore the problem of obtaining "exact" formulas for $r(n;\ f)$ is solved completely. If the genus contains more than one class, then it is necessary to determine the cusp form $X(\tau)$. Many papers are devoted to the problem of finding $X(\tau)$. The cusp forms in these works are constructed in the form of linear combinations of product of simple theta-functions with characteristics or their derivatives [8], products of Jacobi theta-functions or their derivatives [7], and theta-functions with spherical polynomials [8]. All these functions are special cases of the generalized theta-functions with characteristics defined by (1).

In the present paper, using modular properties of these functions, a new cusp form of weight $\frac{9}{2}$ is constructed which belongs to the space of entire modular forms of type $\left(-\frac{9}{2},\ 48,\ v(M)\right)$. Here

$$M\in\Gamma_0(48)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(Z)\ \mid\ c\equiv 0\ \ (\mathrm{mod}\,48)\right\}$$

and $v(M)$ is a system of multipliers with respect to the quadratic form of level 16:

$$f^{(k)}=4\sum_{j=1}^{k}x_j^2+\sum_{j=k+1}^{9}x_j^2,\ (k=1,\ 2,\ldots,8).$$

This cusp form can be used to obtain exact formulas for the number of representations of positive integers by quadratic forms in nine variables.

2. Preliminaries

Lemma 1 ([1], p. 444). Let $f=f(x)$ be a positive quadratic form in *s* variables, *N* be the level of *f*. Then function (2) belongs to the space of entire modular forms of type $\left(-s/2,\ N,\ v(M)\right)$, where $v(M)$ is a system of multipliers with respect to *f* and

$$M\in\Gamma_0(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(Z)\ \mid\ c\equiv 0\ \ (\mathrm{mod}\,N)\right\}.$$

Lemma 2 ([1], Lemma 4). Let *k* be an arbitrary vector and *l* be *a* special vector with respect to the form *f*. Then the following equalities hold:

$$\vartheta_{g+N\,k,\,h}\left(\tau;\ P_\nu,\ f\right)=(-1)^{\frac{h'\,A\,k}{N}}\,\vartheta_{g\,h}(\tau;\ P_\nu,\ f),$$

$$\vartheta_{g,\,h+2l}\left(\tau;\ P_\nu,\ f\right)=\vartheta_{g\,h}(\tau;\ P_\nu,\ f).$$

Lemma 3 ([1], p. 444). Let $f_k=f_k(x)$ be a positive quadratic form in *s* variables, A_k be the matrix of f_k, Δ_k be the determinant of A_k, N_k be the level of f_k, $g^{(k)}$ and $h^{(k)}$ be special vectors with respect to A_k (if $2 \nmid \frac{N}{N_k}$, then $h^{(k)}$ is a vector with even components), B_k be arbitrary complex numbers $(k=1,\ 2,\ \ldots,\ j)$. Let $P_\nu^{(k)} = P_\nu^{(k)}(x)$ be a spherical function of order ν with respect to f_k $(k=1,\ 2,\ \ldots,\ j)$ and Δ be the determinant of a positive definite quadratic form f in $s+2\nu$ variables.

Then the function

$$X(\tau;\ f) = \sum_{k=1}^{j} B_k \vartheta_{g^{(k)}h^{(k)}}(\tau;\ P_\nu^{(k)},\ f_k)$$

is an entire modular form of type $(-s/2,\ N,\ v(M))$ if and only if the following conditions are met

$$N_k \,\Big|\, N,\ N_k^2 \,\Big|\, f_k(g^{(k)}),\ 4N_k \,\Big|\, \frac{N}{N_k} f_k(h^{(k)}) \tag{3}$$

and for all α and δ such that $\alpha\delta \equiv 1 \pmod{N}$ we have

$$\sum_{k=1}^{j} B_k \vartheta_{\alpha g^{(k)}h^{(k)}}(\tau;\ P_\nu^{(k)},\ f_k)(\operatorname{sgn}\delta)^\nu \left(\frac{(-1)^{\left[\frac{s}{2}\right]}\Delta_k}{|\delta|}\right) = \left(\frac{(-1)^{\frac{[s+2\nu]}{2}}\Delta}{|\delta|}\right) \sum_{k=1}^{j} B_k \vartheta_{g^{(k)}h^{(k)}}(\tau;\ P_\nu^{(k)},\ f_k).$$

Lemma 4 ([1], p. 446). If all conditions of Lemma 3 are fulfilled and $\nu > 0$, then the function $X(\tau)$ is a cusp form of type $\left(-\left(\frac{s}{2}+\nu\right),\ N,\ v(M)\right)$.

3. Basic result

Theorem. *Assume* $k = 1,\ 2,\ \ldots,\ 8$,

$$f^{(k)} = 4\sum_{j=1}^{k} x_j^2 + \sum_{j=k+1}^{9} x_j^2,\ P_3 = x_1x_2x_3,\ h' = (0,\ 0,\ 0)$$

$(g^{(1)})' = (8,\ 8,\ 8)$, $(g^{(2)})' = (16,\ 16,\ 16)$, $(g^{(3)})' = (8,\ 8,\ 16)$, $(g^{(4)})' = (16,\ 16,\ 8)$, $f_1 = 12x_1^2 + 12x_2^2 + 12x_3^2$.

Let $B_1,\ B_2,\ B_3,\ B_4$ *be arbitrary complex numbers. Then the function*

$$X(\tau) = B_1\vartheta_{g^{(1)}h}(\tau;\ P_3,\ f_1) + B_2\vartheta_{g^{(2)}h}(\tau;\ P_3,\ f_1) + B_3\vartheta_{g^{(3)}h}(\tau;\ P_3,\ f_1) + B_4\vartheta_{g^{(4)}h}(\tau;\ P_3,\ f_1)$$

is a cusp form of type $\left(-\frac{9}{2},\ 48,\ v(M)\right)$, *where* $v(M)$ *is a system of multipliers with respect to* $f^{(k)}$ *(all of these forms have a same system of multipliers).*

Proof. In Lemma 3 we assume

$\Delta = 2^9 \cdot 4^k \quad k = 1,\ 2,\ \ldots,\ 8$,

$f = f^{(k)},\ N_1 = 48,\ N_2 = 48,\ N_3 = 48,\ N_4 = 48$,

$\nu = 3,\ P_3^{(1)} = P_3^{(2)} = P_3^{(3)} = P_3^{(4)} = P_3 = x_1x_2x_3$,

$N = 48,\ \Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 6^3 \cdot 4^3$,

$(g^{(1)})' = (8,\ 8,\ 8)$, $(g^{(2)})' = (16,\ 16,\ 16)$, $(g^{(3)})' = (8,\ 8,\ 16)$,

$g^{(4)}$ $(16,\ 16,\ 8)$, $j=4,\ s=3$.

It is easy to verify that the function $X(\tau)$ satisfies the conditions (3) of Lemma 3.

We have

$$(\operatorname{sgn}\delta)^{\nu}\left(\frac{(-1)^{\left[\frac{s}{2}\right]}\Delta_1}{|\delta|}\right)=(\operatorname{sgn}\delta)^{\nu}\left(\frac{(-1)^{\left[\frac{s}{2}\right]}\Delta_2}{|\delta|}\right)=(\operatorname{sgn}\delta)^{\nu}\left(\frac{(-1)^{\left[\frac{s}{2}\right]}\Delta_3}{|\delta|}\right)=$$

$$=(\operatorname{sgn}\delta)^{\nu}\left(\frac{(-1)^{\left[\frac{s}{2}\right]}\Delta_4}{|\delta|}\right)=\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)\left(\frac{2}{|\delta|}\right),$$

$$\left(\frac{(-1)^{\left[\frac{s+2\nu}{2}\right]}\Delta}{|\delta|}\right)=\left(\frac{2}{|\delta|}\right). \tag{4}$$

If $\alpha\delta\equiv 1(\operatorname{mod}48)$, then $\alpha\delta\equiv 1(\operatorname{mod}12)$. In particular,

$\alpha\equiv\pm 1,\pm 5(\operatorname{mod}12)$ and respectively $\delta\equiv\pm 1,\pm 5(\operatorname{mod}12)$.

If in (4) we take $\delta>0$ and $\delta\equiv 1(\operatorname{mod}12)$, then

$$\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)=\left(\frac{-3}{|\delta|}\right)=1.$$

In the case $\delta>0$ and $\delta\equiv -5(\operatorname{mod}12)$, we have

$$\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)=\left(\frac{-3}{|\delta|}\right)=-\left(\frac{3}{|\delta|}\right)=\left(\frac{\delta}{3}\right)=1.$$

But if in (4) we take $\delta<0$ and $\delta\equiv 1(\operatorname{mod}12)$, then

$$\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)=-\left(\frac{-3}{-\delta}\right)=\left(\frac{3}{-\delta}\right)=-\left(\frac{-\delta}{3}\right)=1.$$

In the case $\delta<0$ and $\delta\equiv -5(\operatorname{mod}12)$, we have

$$\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)=-\left(\frac{-3}{-\delta}\right)=-\left(\frac{-\delta}{3}\right)=1.$$

Thus

$$\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)=1 \quad\text{for}\quad \delta\equiv 1(\operatorname{mod}6). \tag{5}$$

Now suppose that in (4) $\delta>0$ and $\delta\equiv 5(\operatorname{mod}12)$. Then

$$\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)=\left(\frac{-3}{\delta}\right)=\left(\frac{3}{\delta}\right)=\left(\frac{\delta}{3}\right)=-1.$$

In the case $\delta>0$ and $\delta\equiv -1(\operatorname{mod}12)$, we have

$$\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)=\left(\frac{-3}{\delta}\right)=-\left(\frac{3}{\delta}\right)=\left(\frac{\delta}{3}\right)=-1.$$

But if in (4) we take $\delta<0$ and $\delta\equiv 5(\operatorname{mod}12)$, then

$$\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)=-\left(\frac{-3}{-\delta}\right)=\left(\frac{3}{-\delta}\right)=-\left(\frac{-\delta}{3}\right)=-1.$$

In case $\delta<0$ and $\delta\equiv -1(\operatorname{mod}12)$, we get

$$\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)=-\left(\frac{-3}{-\delta}\right)=-1.$$

Thus

$$\operatorname{sgn}\delta\left(\frac{-3}{|\delta|}\right)=-1 \ \text{ for } \delta\equiv -1 \ (\operatorname{mod}6). \tag{6}$$

Hence, by (4), (5) and (6), for all α and δ with $\delta\equiv 1(\operatorname{mod}48)$, we have

$$(\operatorname{sgn}\delta)^{\nu}\left(\frac{(-1)^{\left[\frac{s}{2}\right]}\Delta_k}{|\delta|}\right)=$$

$$=\left(\frac{(-1)^{\left[\frac{s+2\nu}{2}\right]}\Delta}{|\delta|}\right) \ \text{ for } \alpha\equiv 1(\operatorname{mod}6) \tag{7}$$

$$=-\left(\frac{(-1)^{\left[\frac{s+2\nu}{2}\right]}\Delta}{|\delta|}\right) \ \text{ for } \alpha\equiv -1(\operatorname{mod}6).$$

Due to Lemma 2

$$\vartheta_{\alpha g^{(1)}h}(\tau;\ P_3,\ f_1)=\vartheta_{g^{(1)}h}(\tau;\ P_3,\ f_1) \ \text{ for } \alpha\equiv 1(\operatorname{mod}6)$$

$$=-\vartheta_{g^{(1)}h}(\tau;\ P_3,\ f_1) \ \text{ for } \alpha\equiv -1(\operatorname{mod}6) \tag{8}$$

$$\vartheta_{\alpha g^{(2)}h}(\tau;\ P_3,\ f_1)=\vartheta_{g^{(2)}h}(\tau;\ P_3,\ f_1) \ \text{ for } \alpha\equiv 1(\operatorname{mod}6)$$

$$=-\vartheta_{g^{(2)}h}(\tau;\ P_3,\ f_1) \ \text{ for } \alpha\equiv -1(\operatorname{mod}6) \tag{9}$$

$$\vartheta_{\alpha g^{(3)}h}(\tau;\ P_3,\ f_1)=\vartheta_{g^{(3)}h}(\tau;\ P_3,\ f_1) \ \text{ for } \alpha\equiv 1(\operatorname{mod}6)$$

$$=-\vartheta_{g^{(3)}h}(\tau;\ P_3,\ f_1) \ \text{ for } \alpha\equiv -1(\operatorname{mod}6) \tag{10}$$

$$\vartheta_{\alpha g^{(4)}h}(\tau;\ P_3,\ f_1)=\vartheta_{g^{(4)}h}(\tau;\ P_3,\ f_1) \ \text{ for } \alpha\equiv 1(\operatorname{mod}6)$$

$$=-\vartheta_{g^{(4)}h}(\tau;\ P_3,\ f_1) \ \text{ for } \alpha\equiv -1(\operatorname{mod}6) \tag{11}$$

By (7) - (11) it is easy to verify that all conditions of Lemma 3 are satisfied and therefore the function $X(\tau)$ belongs to the space of entire modular forms of type $\left(-9/2,\ 48,\ \nu(M)\right)$. By Lemma 4 our theorem is proved.

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