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Mathematics

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Taylor Expansion and Sobolev Spaces

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ABSTRACT. In this paper a new characterization of functions in the Sobolev space $W^{m,p}(\mathbb{R}^n), m \geq 1$, in the form of a pointwise inequality is given. This inequality reveals the local and global *m*-th order polynomial-like behaviour of these functions. *© 2011 Bull. Georg. Natl. Acad. Sci.*

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1. Introduction. Formulation of the theorem

In this note we present some complementary comments and remarks to a series of recent papers on pointwise characterization of Sobolev spaces $W^{m,p}(\Omega)$ in smooth subdomains Ω of $\mathbb{R}^n$ as subspaces of $W^{k,p}(\Omega)$ for some $k < m$ $(W^{0,p}(\Omega) = L^p(\Omega))$. For $k = m-1$ this characterization has been obtained in [1-3] in terms of the Taylor remainder

$$R^{m-1}f(y,x) = f(y) - T_x^{m-1}f(y) = f(y) - \sum_{l=0}^{m-1} f^{(l)}(x)\frac{(y-x)^l}{l!} \tag{1.1}$$

centered at $x \in \Omega$, in the form of the pointwise inequalities

$$|R^{m-1}f(y,x)| \leq |x-y|^m \left(a_f(x) + a_f(y)\right) \tag{1.2}$$

for some function $a_f \in L^p$. The Lipschitz type functional coefficients a_f on the right-hand side of (1.2) are not defined uniquely; for convenience, they are all called mean maximal *m*-gradients of the function f and, roughly speaking, all of them can be majorized by the local maximal function of the generalized Sobolev gradient $|\nabla^m f|$ of *f* in $W^{m,p}(\Omega)$; for details see [1-3].

As is well known, the Taylor remainder $R^{m-1}f(y,x)$ has the finite difference approximation

$$\tilde{\Delta}^m f(x,y) = \Delta^m f(x;h) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(x+jh), \quad h = \frac{y-x}{m}. \tag{1.3}$$

This suggests the following finite difference expression for the pointwise inequality (1.2):

$$| \tilde{\ }^{m} f(x,y)| \quad |x \quad y|^{m} \quad \hat{a}_f(x) \quad \hat{a}_f(y)$$

or

$$\left| \sum_{j \ 0}^{m} (\ 1)^j \binom{m}{j} f(x \quad jh) \right| \quad |x \quad y|^m \quad \hat{a}_f(x) \quad \hat{a}_f(y) \tag{1.4}$$

for some $\hat{a}_f \quad L^p$.

The main result of this note is that the pointwise inequality (1.4) also characterizes the Sobolev spaces $W^{m,p}(\ )$.

For the case m=1, (1.2) and (1.4) coincide, and are meaningful for arbitrary metric measure spaces. Consequences of this fact have been extensively studied by P.Hajłasz, resulting in the development of a theory of Sobolev type spaces $W^{1,p}(X,d,\mu)$, $p \quad 1$ on arbitrary metric measure spaces (X,d,μ) [4, 5].

Another pointwise characterization of the spaces $W^{m,p}(\ )$ was proposed by Brezis, Bourgain and Mironescu [6] (m=1) in the form of the finiteness condition

$$A(f,p)(\ ) \quad \liminf_{\varepsilon \ 0} \quad \frac{|f(y) \quad f(x)|^p}{|x \quad y|^p} \rho_\varepsilon(x \quad y)dxdy \qquad , \tag{1.5}$$

where ρ_ε is a family of radial mollifiers approximating the Dirac mass $\delta(x \quad y)$ at the diagonal $\{x \quad y\}$ in the Cartesian product . The condition (1.5) expresses the fact that the total mass of the difference quotient $R^{m \ 1} f(y,x)/|x \quad y|^m \ (m \quad 1)$ "has a finite p-trace" concentrated on the diagonal . If (1.5) is finite then

$$\lim_{\varepsilon \ 0} \quad \frac{|f(y) \quad f(x)|^p}{|x \quad y|^p} \rho_\varepsilon(x \quad y)dxdy \quad K_{p,n} \quad | \ f|^p \tag{1.6}$$

with some constant $K_{p,n}$ [6]. Thus (1.5) allows one to recover the seminorm $\| \ f\|_{L^p}$ of f in $W^{1,p}$. Obviously, for m=1, (1.2) and (1.4) imply (1.5). In fact, by the result of Brezis-Bourgain-Mironescu [6], (1.5) is equivalent to (1.2).

The generalization of (1.5)-(1.6) to higher order Sobolev spaces, suggested during the last years on several occasions, was actually performed in [7] by replacing the fraction in (1.5) by $R^{m \ 1} f(y,x)/|x \quad y|^m$:

$$\liminf_{\varepsilon \ 0} \quad \frac{|R^{m \ 1} f(y;x)|^p}{|x \quad y|^{mp}} \rho_\varepsilon(x \quad y)dxdy \qquad . \tag{1.7}$$

Also, recently in [8] R. Borghol generalized the Brezis-Bourgain-Mironescu result using the finite difference remainder $\tilde{\ }^{m} f(x;y)$ to characterize the Sobolev space $W^{m,p}(\ )$ by the finiteness condition

$$\liminf_{\varepsilon \ 0} \quad \frac{|\tilde{\ }^{m} f(x;y)|^p}{|x \quad y|^{mp}} \rho_\varepsilon(x \quad y)dxdy \qquad . \tag{1.8}$$

In the papers [7] and [8] analogues of (1.6) are proved as well.

Summarizing the above results, we have

Theorem. *For every function* $f \quad W^{m,p}(\ )$, m *integer,* p>1, *the conditions* (1.2), (1.4), (1.7) *and* (1.8) *hold true. Conversely, any of them characterizes the Sobolev class* $W^{m,p}(\ )$, *and thus they are all equivalent.*

Notice the advantage of the conditions (1.4) and (1.8) over (1.2) and (1.7): they characterize the Sobolev functions as elements of Lebesgue spaces L^p for p>1, i.e. regular distributions as functionals on the sufficiently smooth functions [9, 10].

Let us remark also that all the equivalences named in the theorem may be supplemented by the corresponding equivalent inequalities for natural norms related with each case, with full control of the involved constants.

2. Sketch of the proof of the Theorem

It is not our aim here to present ab ovo the detailed systematic proof of the theorem. This would require a much longer substantial exposition of the foundations of the general theory of Sobolev spaces $W^{m,p}$ in the context of pointwise inequalities and is postponed to a pending collective effort with a group of younger interested colleagues. Instead we want to show that some of the ideas, results and the technical tools used in the published papers [1,2, 6-8,11] referred above are sufficient to justify all conclusions of the theorem.

As a missing link in the chain of results published in [1, 2, 6-8] referred to above, and required for the full proof of the Theorem, we propose the following

Lemma. *For functions f in the Sobolev space* $W^{m,p}(\mathbb{R}^n)$ *the following estimate holds:*

$$|\tilde{\Delta}^m f(x;y) - R^{m-1} f(y;x)| \le |x-y|^m \left(\hat{\hat{a}}_{f,\delta}(x) + \hat{\hat{a}}_{f,\delta}(y)\right) \tag{2.1}$$

for some $\hat{\hat{a}}_{f,\delta}(x) \in L^p(\mathbb{R}^n)$, $\delta \sim |x-y|$.

Actually the functional Lipschitz coefficient in (2.1) can be taken as the Hardy-Littlewood local maximal function of the *m*-th gradient $\nabla^m f \in L^p(\mathbb{R}^n)$ (the Lipschitz type coefficients used here and below differ from case to case; we stress, however, that they all are uniformly controlled by the local maximal function of the highest Sobolev gradient of the considered function *f*). Analogous inequality is true also for functions in subdomains of $\mathbb{R}^n$, with sufficiently smooth boundary.

Because of the embarassing familiarity of the calculations, we present them in detail as direct consequences of the formula for the Taylor remainder in integral form [12], as used in [1, 2, 13, 15]. To make things more transparent, we start from the one-dimensional case (amazingly enough, we have not been able to find in the literature the characterization (1.4) for Sobolev spaces even on the line $\mathbb{R}$).

Proof of the Lemma. By the Taylor formula with remainder in integral form [12] we write

$$f(x+h) - T_x^{m-1} f(x+h) = \frac{1}{(m-1)!}\frac{1}{h}\int_x^{x+h} f^{(m)}(\tau)(x+h-\tau)^{m-1} h\,d\tau . \tag{2.2}$$

Hence

$$\left|f(x+h) - T_x^{m-1} f(x+h)\right| \le |h|^m M_\delta f^{(m)}(x), \quad |h| \le \delta, \tag{2.3}$$

where

$$M_\delta g(x) = \sup_{|h|\le\delta}\frac{1}{|h|}\int_x^{x+h} g(t)dt \tag{2.4}$$

is the original classical Hardy-Littlewood maximal function [14] on the real line.

For $h = \frac{y-x}{m}$, $hm = y-x$, the left hand side of (2.3) may be written in the form

$$f(x+jh) - \sum_{l=0}^{m-1}\frac{f^{(l)}(x)(jh)^l}{l!}. \tag{2.5}$$

Hence

$$\sum_{j=0}^{m-1}(-1)^j\binom{m}{j} f(x+jh) - \sum_{j=0}^{m-1}(-1)^j\sum_{j=0}^{m-1}\frac{f^{(l)}(x)}{l!}h^l j^l\binom{m}{j}$$

$$\sum_{l=0}^{m-1}\frac{f^{(l)}(x)}{l!}h^l\sum_{j=0}^{m-1} j^l\binom{m}{j}(-1)^j = \sum_{l=0}^{m-1}\frac{f^{(l)}(x)}{l!}(hm)^l(-1)^{m-1}, \tag{2.6}$$

where we used the elementary formulas

$$\sum_{j=0}^{m-1}(-1)^j j^l\binom{m}{j} = (-1)^{m-1}m^l, \quad l < m. \tag{2.7}$$

By (2.5) and (2.6) we get ($hm = y - x$)

$$\sum_{j=0}^{m-1}(-1)^j\binom{m}{j} f(x+jh) = \sum_{j=0}^{m-1}(-1)^j\binom{m}{j} f(x+jh) + (-1)^m f(y) - (-1)^m f(y) = (-1)^m R^{m-1}f(y;x) \tag{2.8}$$

or

$$\tilde{\Delta}^m f(y,x) = (-1)^m R^{m-1}f(y;x). \tag{2.9}$$

The calculations (2.5)-(2.8) are performed modulo the right hand side of (2.3). If, instead, (2.3) is used in the form of the precise pointwise estimate, a correction has to be introduced leading to the inequality

$$|\tilde{\Delta}^m f(y,x) - (-1)^m R^{m-1}f(y;x)| \le C(m)|x-y|^m M_\delta f^{(m)}(x), \quad |x-y| \le \delta \tag{2.10}$$

as asserted in the lemma.

For the full discussion in $\mathbb{R}^n$, $n > 1$, it is necessary to introduce the multiindex notation. Then l in (2.4) and (2.5) is the multiindex $l = (l_1,\dots,l_n)$, $l_i \ge 0$,

$$|l| = \sum_{i=1}^{n} l_i,\ l! = l_1!\dots l_n!,\ \left(\frac{\partial}{\partial x}\right)^l = \left(\frac{\partial}{\partial x_1}\right)^{l_1}\dots\left(\frac{\partial}{\partial x_n}\right)^{l_n},\ h^l = h_1^{l_1}\dots h_n^{l_n}.$$

Now $(hj)^l = h^l j^{|l|}$ and the calculations (2.5)-(2.9) go through (for $n>1$) verbatim with (2.7) applied for $l \to |l| < m$. The only problem is to understand what replaces (2.3) in the multidimensional case. But this is answered by the theory of pointwise inequalities of Sobolev type as presented in [1, 2] and many papers and lecture notes by P. Hajłasz afterwards.

Let us recall that the two essential steps in this procedure are: 1) the Sobolev integral representation of the left hand side in (2.3) or the Taylor remainder term $R^{m-1}f(x;h)$ estimate in the form of Riesz potentials and 2) the Hedberg lemma, estimating the Riesz potentials by the multi-dimensional Hardy-Littlewood maximal function.

Let us mention also that the computations on pages 44-48 of [13] are very helpful in this respect.

Proof of the Theorem. After all said above the required proof is rather direct and proceeds by reference to the papers [1, 2, 6-8,]. It can be organized in a sequence of implications described by the table

$$\begin{array}{ccccc} (1.0) & \xrightarrow{[1],[2]} & (1.2) & \Rightarrow & (1.7) \\ & & & & \Downarrow [6],[7] \\ & \text{Lemma} & & & (1.0) \\ & & & & \Downarrow [8] \\ & & (1.4) & \Rightarrow & (1.8) \end{array}$$

where the symbol (1.0) stands for the basic assumptions of the theorem: $f \in W^{m,p}(\mathbb{R}^n)$ and the implication arrows are marked by our reference list numbers. The unmarked arrows describe simple direct conclusions. □

It is worthwhile to stress the fact, maybe as yet not sufficiently clearly exposed in the literature, that all these operations (Riesz potentials, maximal functions) and, in consequence, pointwise inequalities are invariant under convolutions, or taking Steklov means. This remark allows us to reduce the proofs to the case of smooth or Lipschitz functions, considerably simplifying the exposition. In this connection the advantage of the pointwise inequalities (1.4) over (1.7) and (1.2) (m>1) appears again.

3. General comments and conclusions

The most subtle and technically delicate of the presented four types of characterizations of Sobolev spaces is undoubtedly the Brezis's type characterization (1.5)-(1.7), references [6, 7]. The most explicit (and "visible") seems to be the one described by inequalities (1.2) and (1.4). In this context our theorem can be interpreted as a ''Tauberian'' type theorem.

Pointwise description of properties of Sobolev functions appears in the recent literature at several places. Besides the above-quoted papers let us also point out [15]. Leaving aside for the time being the task to reviewing all theses cases, it seems that the number of their applications grows noticeably: e.g. Lusin's type approximation problems for Sobolev functions by functions from Lipschitz-Zygmund classes, continuously differentiable functions in the sense of Whitney etc. [16, 17], various problems of harmonic analysis, higher order Zygmund's "smooth functions" [16].

Pointwise inequalities expose also the well-known fact – Nikolski's papers, quoted in [13] – of characterizing Sobolev functions by their behaviour on lines or general k-codimensional hyperplanes of $\mathbb{R}^n$.

It has been stressed in the literature on several occasions that the "distributional" – defined by the duality type argument – character of functional analytical definition of Sobolev spaces does not serve well the needs of non-linear partial differential equations. It may be that systematic pointwise approach will be helpful in this respect.

Pointwise inequalities (1.2) and (1.4) admit a direct geometric interpretation: the remainders $R^{m\ 1} f(y;x)$ and $\tilde{\ }^{m} f(y,x)$ measure the deviations of the function $f(x)$ from polynomials of order less than m at the point x. Conditions $R^{m\ 1} f(y,x) \quad 0$ or $\tilde{\ }^{m} f(y,x) \quad 0$ in a neighbourhood of x are equations for the polynomials of order less or equal m–1. Inequalities (1.2) or (1.4) and the theorem give the "simple" and "precise" estimates for these deviations for functions f in the Sobolev class $W^{m,p}$.

Another, though closely related, way of interpretation of the inequalities (1.2)-(1.4) and the theorem is to look at them in the context of interpolation-extrapolation relation: given the inequalities (1.2), (1.4) as information about the "interpolation data", the theorem allows us to deduce the global information – "extrapolation conclusions" – about the (global) behaviour of the function f as belonging to the Sobolev class $W^{m,p}$ and vice versa.

This somehow relates the mentioned problems with the series of voluminous and imposing research program of C. Fefferman (see [18, 19]and other referred to therein) on the structure of an arbitrary smooth function on subsets of $\mathbb{R}^n$ (classes $C^k(\ ),\ k \quad 0)$. On the other hand, this research program of C. Fefferman has also many points of contact with the study of the structure of approximately differentiable functions and Lusin-Whitney type properties described in [20].

At last, in contrast to the above-sketched "constructive" proof of the inequalities (1.4), in the spirit of early Sobolev's works, let us mention also a "soft" proof referring to the first applications of general functional analysis of Banach spaces to "concrete" problems of "classical" analysis.

For this purpose let us introduce the space $\hat{W}^{m,p}$ as the linear subset of $L^p(\mathbb{R}^n), \qquad p \quad 1$ satisfying the condition (1.4). By the result of [8] $\hat{W}^{m,p}$ is a subspace of $W^{m,p}$. Moreover it is a closed subspace of $W^{m,p}$ and the semi-norm of f as an element of $\hat{W}^{m,p}$ is estimated by the L^p-norm of the "generalized" Lipschitz modulus $\hat{a}_f$. On the other hand, obviously, the space C_0 of smooth compactly supported functions is a dense subspace of $W^{m,p}$, $C_0 \quad \hat{W}^{m,p} \quad W^{m,p}$. Hence, by completeness of Banach spaces, $\hat{W}^{m,p} \quad W^{m,p}$ and (1.4) follows.

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