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Mathematics

Ito's Formula in a Banach Space

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ABSTRACT. Ito's formula for the generalized random processes and for the random processes with values in a separable Banach space is proved. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *Wiener processes, generalized random processes, Banach space valued random processes, Ito's formula.*

As in the finite dimensional case, the Ito formula plays an important role in the infinite dimensional stochastic analysis. For the cases when the Banach spaces have special geometrical properties, the Ito formula was proved in [1], [2]. For the Wiener process in an arbitrary separable Banach space, the Ito formula was proved in [3]. The main problem to prove this formula for nonanticipating functions with values in an arbitrary Banach space is to find such a sequence of step functions converging to the integrand function that their stochastic integrals converge to the stochastic integral from the integrand function. It turns out that it is possible to weaken the convergence to prove the Ito formula. As we use the concept of the generalized random element, and the Ito formula will be useful for the generalized random processes, we prove the Ito formula first for the generalized random processes and as a consequence for the Banach space valued random process.

Let X be a real separable Banach space, X^* - its conjugate, $B(X)$ - the Borel σ-algebra of X, (Ω, B, P) - a probability space. The continuous linear operator $T : X^* \to L_2(\Omega, B, P)$ is called a generalized random element (GRE). Denote by $M_1 := L(X^*, L_2(\Omega, B, P))$ the Banach space of GRE with the norm $\|T\|_{M_1} := \sup_{\|x^*\|\le 1}\|Tx^*\|_{L_2}$. A random element (measurable map) $\xi : \Omega \to X$ is said to have a weak second order, if for all $x^* \in X^*$, $E\left\langle \xi, x^*\right\rangle^2 < \infty$. We can realize the random element ξ as an element of $M_1 : T_\xi x^* = \left\langle \xi, x^*\right\rangle$ (Continuity of T_ξ follows from the closed graph theorem). Denote by M_2 the linear space of all random elements of the weak second order with the norm $\|\xi\| = \|T_\xi\|$. Thus, we can assume $M_2 \subset M_1$.

Let $(W_t)_{t\in[0,1]}$ be the one-dimensional Wiener process, $(F_t)_{t\in[0,1]}$ an increasing family of σ-algebras such that a) W_t is F_t-measurable for all $t\in[0,1]$; b) W_s-W_t-is independent of the σ-algebra F_t for all $s>t$. F_t contains all P-null sets in B. Let $(T_t)_{t\in[0,1]}$ be a family of GRE. We call it a generalized random process (GRP). If we have a weak second order random process $(\xi_t)_{t\in[0,1]}$, $\xi_t : \Omega \to X$, it will be realized as a GRP: $T_{\xi_t} x^* = \left\langle \xi_t, x^*\right\rangle$.

Definition 1. The GRP $(T_t)_{t\in[0,1]}$ is called nonanticipating with respect to $(F_t)_{t\in[0,1]}$, if for all $x^* \in X^*$ the real random process $T_t x^*$ has a stochastically equivalent (t,w)- measurable modification and for all $t\in[0,1]$ $T_t x^*$ is F_t-measurable.

By TM_1 we denote the linear normed space of nonanticipating GRP $(T_t)_{t\in[0,1]}$, for which $\|T_t\| = \sup_{\|x^*\|\le 1}(\int_0^1 E(T_t x^*)^2 dt)^{\frac{1}{2}} < \infty$. For all $x^* \in X^*$ we can define the scalar stochastic integral.

Definition 2. For any $(T_t)_{t\in[0,1]} \in TM_1$, the operator $\mathrm{I}(T)x^* = \int_0^1 T_t x^* dW_t$ is called the generalized stochastic integral (GSI) of $(T_t)_{t\in[0,1]}$.

It is easy to see that $\mathrm{I}(T)\in M_1$ and $\mathrm{I}: TM_1 \to M_1$ is an isometric operator.

Lemma 1. *Let the nonanticipating* GRP $(T_t)_{t\in[0,1]}$ *be such that* $T:[0,1]\to M_1$ *is separable valued and* $\int_0^1 \|T(t)\|_{M_1}^2 dt < \infty$, *then there exists the sequence of nonanticipating step functions* $T_n:[0,1]\to M_1$ *such that* $\int_0^1 \|T(t)-T_n(t)\|_{M_1}^2 dt \to 0$ *and* $\|I(T)-I(T_n)\|_{M_1} \to 0$.

Proof. It is easy to see that we can assume that T is bounded. If $T:[0,1]\to M_1$ is continuous, then $\left\|T(t)-\sum_{k=0}^{n-1} T(\frac{k}{n}) I_{(\frac{k}{n},\frac{k+1}{n}]}\right\|_{M_1} \to 0$, when $n\to\infty$. As $\left\|\sum_{k=0}^{n-1} T(\frac{k}{n}) I_{(\frac{k}{n},\frac{k+1}{n}]}\right\|_{M_1} \le C$, by the Lebesgue theorem, $\int_0^1 \|T(t)-T_n(t)\|_{M_1}^2 dt \to 0$, where $T_n(t) = \sum_{k=0}^{n-1} T(\frac{k}{n}) I_{(\frac{k}{n},\frac{k+1}{n}]}$. In this case the sequence $(T_n(t))_{n\in N}$ satisfies the conditions of the lemma. Let now $T(t)$ be arbitrary. For all $x^*\in X^*$ and $g\in L_2(\Omega,B,P)$, $\left\langle T(t)x^*, g\right\rangle_{L_2}$ is measurable and linear bounded functionals $\Gamma := \{f(x^*,g): M_1 \to R,\ \left\langle f,T\right\rangle = \left\langle Tx^*, g\right\rangle_{L_2}\}$ are total in M_1^*, therefore, as $T:[0,1]\to M_1$ is separable valued, by the Pettis theorem (see [4] prop.1.1.10) it follows that T is measurable. Hence, the Bochner integral $\int_s^t T(u)du$ exists. Let $T_m(t) = m\int_{(t-\frac{1}{m})\vee 0}^{t} T(s)ds$. $T_m(t)\to T(t)$ a.s. when $m\to\infty$ (see [5] corr.2 Theor.3.8.5). By the Lebesgue theorem, $\int_0^1 \|T_m(t)-T(t)\|_{M_1}{}^2 dt \to 0$. As $T_m(t)$ is continuous, there exists the sequence of nonanticipating step functions $T_{m,n}$, n=1,2,... such that $\int_0^1 \|T_m - T_{m,n}\|_{M_1}^2 dt \to 0$. Consequently, we can find the sequence of nonanticipating step functions T_n, n=1,2,... such that $\int_0^1 \|T_n - T\|_{M_1}^2 dt \to 0$. Then $\|I(T_n)-I(T)\|_{M_1} \to 0$.

Remark. If $(T_t)_{t\in[0,1]}$ is such that $T_t \in \overline{M}_2$, then the step functions T_n in the lemma 1 can be found X valued.

Definition 3. A random element $\xi:\Omega\to X$ is called the stochastic integral of $(T_t)_{t\in[0,1]}$ (if such an element exists), if $\left\langle \xi, x^*\right\rangle = I(T)x^*$ a.s. and denote $\xi := \int_0^1 T_t dW_t$.

Definition 4. A nonanticipating GRP is called the generalized Ito process, if there exist nonanticipating GRP $a=(a_t)_{t\in[0,1]}$ and $b=(b_t)_{t\in[0,1]}$ such that, for all $x^*\in X^*$,

$$T_t x^* = T_0 x^* + \int_0^t a(s)x^* ds + \int_0^t b(s)x^* dW_s \quad \text{a.s..}$$

Lemma 2. Let the Ito process $T_t x^* = T_0 x^* + \int_0^t a(s)x^* ds + \int_0^t b(s)x^* dW_s$ be such that $a:[0,1]\to M_1$ and $b:[0,1]\to M_1$ are separable valued and $\int_0^1 \|a(t)\|_{M_1}{}^2 dt < \infty$, $\int_0^1 \|b(t)\|_{M_1}{}^2 dt < \infty$. Then there exist the sequences of nonanticipating step functions $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ such that $\int_0^1 \|a_n(t)-a(t)\|_{M_1}^2 dt \to 0$, $\int_0^1 \|b_n(t)-b(t)\|_{M_1}^2 dt \to \infty$ and $\left\|T^{(n)}{}_t - T_t\right\|_{M_1}^2 \to 0$ uniformly for t, where

$$T^{(n)}{}_t x^* = T_0 x^* + \int_0^t a_n(s)x^* ds + \int_0^t b_n(s)x^* dW_s$$

Proof. Using the lemma 1, we can find nonanticipating $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ such that $\int_0^1 \|a_n(t)-a(t)\|_{M_1}^2 dt \to 0$, $\int_0^1 \|b_n(t)-b(t)\|_{M_1}^2 dt \to 0$. Then

$$\left\|T^{(n)}{}_t - T_t\right\|_{M_1}{}^2 = \sup_{\|x^*\|\le 1} E\Big(\int_0^t (a_n(s)-a(s))x^* ds + \int (b_n(s)-b(s))x^* dW(s)\Big)^2 \le$$

$$2\sup_{\|x^*\|\le 1} E\Big(\int_0^t (a_n(s)-a(s))x^* ds\Big)^2 + 2\sup_{\|x^*\|\le 1} E\Big(\int_0^t (b_n(s)-b(s))x^* dW(s)\Big)^2 \le$$

$$2\sup_{\|x^*\|\le 1} E\int_0^1 ((a_n(s)-a(s))x^*)^2 ds + 2\sup_{\|x^*\|\le 1} E\int_0^1 ((b_n(s)-b(s))x^*)^2 ds \le$$

$$2\int_0^1 \|a_n(t)-a(t)\|_{M_1}{}^2 dt \to 0 + 2\int_0^1 \|b_n(t)-b(t)\|_{M_1}{}^2 dt \to 0 .$$

Theorem 1(Ito's Formula). *Let* $T_t x^* = T_0 x^* + \int_0^t a(s)x^* ds + \int_0^t b(s)x^* dW_s$ *be a generalized Ito process, where a and b are separable valued nonparticipating* GRP *such that* $\int_0^1 \|a(t)\|_{M_1}^2 dt < \infty$, $\int_0^1 \|b(t)\|_{M_1}^2 dt < \infty$. *Let* $f:[0,1]\times M_1 \to M_1$ *be such that the derivatives* $f_t':[0,1]\times M_1 \to M_1$, $f_T':[0,1]\times M_1 \to L(M_1,M_1)$ *and* $f_{T,T}'':[0,1]\times M_1 \to L(M_1, L(M_1,M_1))$ *are continuous. Then*

$$f(t,T_t) = f(0,T_0) + \int_0^t f_t'(s,T_s)ds + \int_0^t f_T'(s,T_s)a(s)ds + \tfrac{1}{2}\int_0^t f_{T,T}''(s,T_s)b(s)b(s)ds + \int_0^t f_T{}'(s,T_s)b(s)dW(s) .$$

Proof. At first we show that it is enough to prove this Theorem for the step functions a and b. Indeed, let $a_n(s)$ and $b_n(s)$ be the sequences of step functions such that $\int_0^1 \|a_n(t)-a(t)\|_{M_1}^2 dt \to 0$ and $\int_0^1 \|b_n(t)-b(t)\|_{M_1}^2 dt \to 0$, then, by the Lemma 2 , $\left\|T^{(n)}{}_t - T_t\right\|^2_{M_1} \to 0$ uniformly for *t*, where

$$T_t^{(n)}x^* = T_0x^* + \int_0^t a_n(s)x^*ds + \int_0^t b_n(s)x^*dW_s \ .$$

Let the Ito formula be true for the step functions:

$$f(t,T_t^{(n)}) = f(0,T_0) + \int_0^t f_t'(s,T_s^{(n)})ds + \int_0^t f_T'(s,T_s^{(n)})a_n(s)ds +$$

$$+\tfrac{1}{2}\int_0^t f''_{T,T}(s,T_s^{(n)})b_n(s)b_n(s)ds + \int_0^t f_T{}'(s,T_s^{(n)})b_n(s)dW(s) \ .$$

As $f_t{}'(s,T_s^{(n)})$ are continuous functions on [0,1] converging to the continuous function $f_t{}'(s,T_s)$, they are bounded. Thereby, by the Lebesgue theorem, we have a convergence $\int_0^t f_t'(s,T_s^{(n)})ds \to \int_0^t f_t'(s,T_s)ds$. Further, we have

$$\int_0^t \left\|f_T{}'(s,T^{(n)}{}_s)a_n(s) - f_T{}'(s,T_s)a_s\right\|ds \le \int_0^1 \left\|f_T{}'(s,T_s^{(n)})\right\|\left\|a_n(s)-a(s)\right\|ds + \int_0^1 \left\|a(s)\right\|\left\|f_T{}'(s,T_s^{(n)}) - f_T{}'(s,T_s)\right\|ds \le$$

$$\le C_1\int_0^1 \left\|a_n(s)-a(s)\right\|ds + \left(\int_0^1 \left\|a(s)\right\|^2 ds\right)^{1/2}\left(\int_0^1 \left\|f_T{}'(s,T_s^{(n)}) - f_T{}'(s,T_s)\right\|^2 ds)\right)^{1/2} \to 0;$$

In principle similarly we can prove $\frac{1}{2}\int_0^t f''_{T,T}(s,T_s^{(n)})b_n(s)b_n(s)ds \to \frac{1}{2}\int_0^t f''_{T,T}(s,T_s)b(s)b(s)ds$ and $\int_0^t f_T{}'(s,T^{(n)}{}_s)b_n(s)dW(s) \to \int_0^t f_T{}'(s,T_s)b(s)dW(s)$. Therefore, it is enough to prove the Ito lemma for the step functions and, by additivity of integrals, we need to prove it when $T_t = T_0 + at + bW_t$, where *a* and *b* are the elements of M_1. For simplicity, we can assume that T_0=0. Then the function $u(t,bW_t) = f(t,at+bW_t)$ has the same smoothness as *f* and so, it is enough to prove the Ito formula for function $u(t,bW_t)$. Let $l = [2^n t]$, $\Delta W = W_{\frac{k}{2^n}} - W_{\frac{(k-1)}{2^n}}$, $\Delta = \frac{1}{2^n}$, $n = 1,2,\dots$. Then, by the Taylor's formula, we have

$$u(t,bW_t) - u(0,0) = \sum_{k\le l} u(\frac{k}{2^n},bW_{\frac{k}{2^n}}) - u(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}}) + u(t,bW_t) - u(\frac{l}{2^n},bW_{\frac{l}{2^n}}) =$$

$$= \sum_{k\le l}\int_0^1 u_t'(\frac{k-1}{2^n} + \frac{s}{2^n},bW_{\frac{k-1}{2^n}})\cdot\frac{1}{2^n}ds + \sum_{k\le l} u_T'(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})b\Delta W +$$

$$+\sum_{k\le l}\int_0^1 (1-s)u''_{TT}(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}} + sb\Delta W)bb\Delta W\Delta W ds + u(t,bW_t) - u(\frac{l}{2^n},bW_{\frac{l}{2^n}}) =$$

$$= \sum_{k\le l} u_t'(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})\cdot\frac{1}{2^n} + \sum_{k\le l}\int_0^1 [u_t'(\frac{k-1}{2^n} + \frac{s}{2^n},bW_{\frac{k-1}{2^n}}) - u_t'(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})]\cdot\frac{1}{2^n}ds +$$

$$+\sum_{k\le l} u_T'(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})b\Delta W + \frac{1}{2}\sum_{k\le l} u''_{TT}(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})]\cdot bb(\Delta W)^2 +$$

$$+\sum_{k\le l}\int_0^1 (1-s)[u''_{TT}(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}} + bs\Delta W) - u''_{TT}(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})]bb(\Delta W)^2 ds +$$

$$+u(t,bW_t)-u(\frac{l}{2^n},bW_{\frac{l}{2^n}})=\sum_{k\le l}u'_t(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})\cdot\frac{1}{2^n}+\sum_{k\le l}u'_T(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})b\Delta W+$$

$$+\frac{1}{2}\sum_{k\le l}u''_{TT}(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})bb(\Delta W)^2+A_n+B_n+\delta_n\text{, where}$$

$$A_n=\sum_{k\le l}\int_0^1[u'_t(\frac{k-1}{2^n}+\frac{s}{2^n},bW_{\frac{k-1}{2^n}})-u'_t(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})]\cdot\frac{1}{2^n}ds,$$

$$B_n=\sum_{k\le l}\int_0^1(1-s)[u''_{TT}(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}}+bs\Delta W)-u''_{TT}(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})]bb(\Delta W)^2ds$$

$$\delta_n=u(t,bW_t)-u(\frac{l}{2^n},bW_{\frac{l}{2^n}}).$$

Denote by $\alpha_n:=\sup_{k\le l}\left\|u'_t(\frac{k-1}{2^n}+\frac{s}{2^n},bW_{\frac{k-1}{2^n}})-u'_t(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})\right\|_{M_1}$ and

$$\beta_n:=\sup_{k\le l}\left\|\int_0^1(1-s)[u''_{TT}(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}}+s\Delta W)-u''_{TT}(\frac{k-1}{2^n},bW_{\frac{k-1}{2^n}})]bbds\right\|_{M_1}.$$

By continuity of u, u'_t, u'_T and u''_{TT} we have the convergence of δ_n, α_n and β_n to 0 when $n\to\infty$. Therefore, $\|A_n\|_{M_1}\le\alpha_n\to 0$, $\|B_n\|_{M_1}\le\beta_n\cdot\sum_{k=1}^{2^n}(W_{\frac{k}{2^n}}-W_{\frac{k-1}{2^n}})^2=\beta_n t\to 0$.

Now, if we act $x^*\in X^*$ to both sides of the equality (1) and pass to the limit in probability, using the one-dimensional technique, we get

$$u(t,bW_t)x^*=u(0,0)x^*+\int_0^t u'_t(s,bW_s)x^*ds+\frac{1}{2}\int_0^t u''_{TT}(s,bW_s)bbx^*ds+\int_0^t u'_T(s,bW_s)bx^*dW_s.$$

Let us now return to the function f and remember that $f(t,at+bW_t)=u(t,bW_t)$, then $u'_t(t,bW)_t=f'_t(t,at+bW_t)+f'_T(t,at+bW_t)a$. Therefore, we have

$$f(t,T_t)=f(0,T_0)+\int_0^t f'_t(s,T_s)ds+\int_0^t f'_T(s,T_s)a(s)ds+\frac{1}{2}\int_0^t f''_{TT}(s,T_s)b(s)b(s)ds+\int_0^t f'_T(s,T_s)b(s)dW.$$

Let the generalized Ito process $T_tx^*=T_0x^*+\int_0^t a(s)x^*ds+\int_0^t b(s)x^*dW_s$ be such that there exists the X-valued process $(\xi_t)_{t\in[0,1]}$ with property $\langle\xi_t,x^*\rangle=T_tx^*$ for all $x^*\in X$. That is, $\xi_t=\xi_0+\int_0^t a(t,\omega)dt+\int_0^t b(t,\omega)dW_t$, where $a:[0,1]\times\Omega\to X$, $b:[0,1]\times\Omega\to X$ are (t,ω)-measurable, F_t-adapted and $\int_\Omega\int_0^t\|a(t,\omega)\|^2dPdt<\infty$, $\int_\Omega\int_0^t\|b(t,\omega)\|^2dPdt<\infty$. Let also $f:[0,1]\times X\to X$, be such that $f:[0,1]\times M_1\to M_1$, f'_t, f'_T and f''_{TT} are continuous by the norm of M_1. Then, taking into consideration the remark of the Lemma 1, we have

$$\langle f(t,\xi_t),x^*\rangle=\langle f(0,\xi_0),x^*\rangle+\left\langle\int_0^t f'_t(s,\xi_s)ds,x^*\right\rangle+$$

$$+\left\langle \int_0^t f_T'(s,\xi_s)a(s)ds, x^*\right\rangle + \left\langle \frac{1}{2}\int_0^t f_{TT}''(s,\xi_s)b(s)b(s)ds, x^*\right\rangle + \int_0^t \left\langle f_T'(s,\xi_s)b(s), x^*\right\rangle dW_s.$$

The first five members of the afore-mentioned equality are functionals from the X-valued processes. Therefore, the stochastic integral $\int_0^1 f_t'(s,\xi_s)b(s)dW_s$ as the X-valued random process exists. Consequently, we have received the Ito formula for the Banach space valued Ito processes.

Theorem 2. *Let* $\xi_t = \xi_0 + \int_0^t a(s,\omega)ds + \int_0^t b(s,\omega)dW_s$ *, where* $a:[0,1]\times\Omega \to X$ *,* $b:[0,1]\times\Omega \to X$ *be* (t,ω) *measurable,* F_t*-adapted and* $\int_\Omega\int_0^t \|a(t,\omega)\|^2 dPdt < \infty$*,* $\int_\Omega\int_0^t \|b(t,\omega)\|^2 dPdt < \infty$*. Let* $f:[0,1]\times X \to X$ *be such that* $f:[0,1]\times M_1 \to M_1$*,* f_t'*,* f_T' *and* f_{TT}'' *are continuous, then*

$$f(t,\xi_t) = f(0,\xi_0) + \int_0^t f_t'(s,\xi_s)ds + \int_0^t f_T'(s,\xi_s)a(s)ds + \frac{1}{2}\int_0^t f_{TT}''(s,\xi_s)b(s)b(s)ds + \int_0^t f_T'(s,\xi_s)b(s)dW_s.$$

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მათემატიკა

იტოს ფორმულა ბანახის სივრცეში

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