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Mathematics

Nonclassical Problems with Nonlocal Initial Conditions for Abstract Second-Order Evolution Equations

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ABSTRACT. The nonclassical problems for abstract second-order evolution equations with nonlocal initial conditions are considered. The existence and uniqueness results for general nonlocal in time problems are proved in suitable spaces of vector-valued distributions with values in abstract Hilbert spaces. An iteration algorithm of approximation of solution to nonclassical problem by a sequence of solutions to corresponding classical problems is constructed and investigated. Applying general results obtained for nonclassical problems in abstract Hilbert spaces, nonlocal in time initial-boundary value problems for hyperbolic equations and systems are studied. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *abstract second-order evolution equations, nonlocal initial conditions, initial-boundary value problems for hyperbolic equations and systems.*

Nonclassical initial-boundary value problems for partial differential equations with nonlocal initial conditions arise in various fields, particularly, in metereology, since long-term weather forecasting is more reliable if time-averaged data rather than the classical initial ones are used [1]. Nonlocal initial conditions are used for investigation of radionuclides propagation in Stokes fluid and problems of predicting geophysical fields [2,3]. Such type initial-boundary value problems, where instead of classical initial conditions a dependence of values of the unknown function at various time moments is given, are called nonlocal in time problems. These problems are generalizations of classical and periodical with respect to the time variable initial-boundary value problems and can be also considered as problems of controllability by initial conditions. Nonlocal in time problem with particular discrete nonlocal initial condition for parabolic equation was studied in [4]. Later on, initial-boundary value problems with nonlocal initial conditions were investigated for various time-dependent partial differential equations and systems [5-7].

In the present paper we investigate nonclassical problems for abstract second-order evolution equations with initial conditions involving general nonlocal operators. Note that the papers, where nonlocal in time problems for hyperbolic equations are considered, mainly deal with specific nonlocal initial conditions. We obtain results on the existence and uniqueness of solutions to the nonclassical problems for second-order evolution equation in abstract Hilbert spaces with general nonlocal initial conditions. We construct an algorithm of approximation of solution of the nonclassical problem by a sequence of solutions to corresponding classical problems and investigate the convergence of the constructed algorithm. On the basis of the obtained general results we study nonlocal in time problems for hyperbolic partial differential equations and systems.

Let V and H be separable Hilbert spaces, V is dense in H and continuously embedded in it. We identify the space H with its dual by using the scalar product $(.,.)_H$ in H and hence we have $V \subset H \subset V'$ with continuous and dense embeddings, where V' is the dual space of V. For Banach spaces X and Y, we denote by $L(X;Y)$ the space of linear continuous operators from X to Y. $C^0([0,T];X)$ is the space of continuous vector-functions on $[0,T]$ with values in X, by $L^2(0,T;X)$ we denote the space of such measurable vector-functions $g:(0,T)\to X$ that $\|g\|_X \in L^2(0,T)$. We identify each vector-function $g\in L^2(0,T;X)$ with a vector-valued distribution on $(0,T)$ ranging in X, and we denote its derivative in the sense of distributions by $g' = dg/dt \in D'(0,T;X) = L(D(0,T);X)$ [8], where $D(0,T)$ is the space of infinitely differentiable functions with compact support in $(0,T)$. We denote by $\mathbf{R}^p$, $p\in\mathbf{N}$, the p-dimensional space of vectors $\vec{x}=(x_k)_{k=1}^p$ with the scalar product $(\vec{x},\vec{y})_{\mathbf{R}^p} = \sum_{j=1}^p x_j y_j$, and with the norm $\|\vec{x}\|^2_{\mathbf{R}^p} = (\vec{x},\vec{x})_{\mathbf{R}^p}$, $\vec{x},\vec{y}\in\mathbf{R}^p$, $\mathbf{R}^{p\times p}$ is the space of $p\times p$ matrices with real entries, and by $\|M\|_{\mathbf{R}^{p\times p}}$ we denote the spectral norm of a matrix $M\in\mathbf{R}^{p\times p}$.

Let $A\in L(V;V')$ be a self-adjoint strongly coercive operator, i.e. the corresponding bilinear form $a(v,w)=\langle Av,w\rangle$ satisfies the following conditions

$$\begin{aligned} &a(v,w)=a(w,v),\ a(v,w)\le c_a\|v\|_V\|w\|_V, \\ &a(v,v)\ge \hat{c}_a\|v\|_V^2,\ c_a,\hat{c}_a = const>0, \end{aligned} \quad \forall\ v,w\in V, \tag{1}$$

where by $\langle .,.\rangle$ we denote the duality relation between the spaces V' and V. Moreover, we assume that the set of eigenvectors $\{v_k\}_{k\in N}$ of the operator A corresponding to the eigenvalues $\{|\lambda_k|^2\}_{k\in N}$ is complete in the space V and orthonormal in H.

We consider nonclassical problem with nonlocal initial conditions for an abstract second-order evolution equation, which admits the following variational formulation: Find the unknown vector-function $u\in C^0([0,T];V)$, $u'\in C^0([0,T];H)$, satisfying the equation

$$\frac{d}{dt}(u'(.),v)_H + a(u(.),v) = (f(.),v)_H,\ \forall v\in V, \tag{2}$$

in the sense of distributions on $(0,T)$ and the following nonlocal initial conditions

$$u(0)=B^1u+B^2u'+u_0,\ u'(0)=B^3u+B^4u'+u_1, \tag{3}$$

where $B^1\in L(C^0([0,T];V);V)$, $B^2\in L(C^0([0,T];H);V)$, $B^3\in L(C^0([0,T];V);H)$ and $B^4\in L(C^0([0,T];H);H)$, u_0, u_1, f are given vector-functions from suitable spaces.

The operators B^1, B^2, B^3 and B^4 in the nonlocal initial conditions (3) define linear continuous functionals $b^1_{jk}(\varphi)=(B^1(\varphi v_k),v_j)$, $b^2_{jk}(\varphi)=(B^2(\varphi v_k),v_j)$, $b^3_{jk}(\varphi)=(B^3(\varphi v_k),v_j)$, $b^4_{jk}(\varphi)=(B^4(\varphi v_k),v_j)$, for all $j,k\in\mathbf{N}$, $\varphi\in C^0([0,T])$, which permits one to construct the operators $\overline{B}^\alpha_p : C^0([0,T];\mathbf{R}^{p\times p})\to\mathbf{R}^{p\times p}$, $p\ge 1$, $\alpha=1,2,3,4$ such that

$$\overline{B}^\alpha_p(F)=G,\ G=(G_{jk}),\ G_{jk}=\sum_{r=1}^p b^\alpha_{jr}(F_{rk}),\ \ \forall F=(F_{rk})\in C^0([0,T];\mathbf{R}^{p\times p}),$$

where $j,k=1,...,p$. Applying the operators $\overline{B}^1_p$, $\overline{B}^2_p$, $\overline{B}^3_p$, $\overline{B}^4_p$ for each invertible diagonal matrix $Q\in\mathbf{R}^{p\times p}$ we define the following matrices

$$\tilde{B}^{1,2}_p(Q)=\overline{B}^1_p(\cos(Qt))-\overline{B}^2_p(\sin(Qt))Q,\quad \hat{B}^{1,2}_p(Q)=-\overline{B}^1_p(\sin(Qt))-\overline{B}^2_p(\cos(Qt))Q,$$

$$\tilde{B}^{3,4}_p(Q)=Q^{-1}(\overline{B}^4_p(\sin(Qt))Q-\overline{B}^3_p(\cos(Qt))),\quad \hat{B}^{3,4}_p(Q)=Q^{-1}(\overline{B}^3_p(\sin(Qt))+\overline{B}^4_p(\cos(Qt))Q).$$

By A_p we denote the diagonal matrix of order p with (j,j)-th entries $|\lambda_j|$, and by $h(A_p)$ we denote the diagonal matrix with entries $h(|\lambda_j|)$ $(1\le j\le p)$, for any analytical function $h:[0,+\infty)\to\mathbf{R}$.

The following theorem is valid for the nonlocal in time problem (2), (3).

Theorem 1. *If* $u_0 \in V$, $u_1 \in H$, $f \in L^2(0,T;H)$, *the nonlocal operators* B^1, B^2, B^3 *and* B^4 *are compact and there exist constants* $\beta = const > 0$ *and* $s \in \mathbf{R}$, $s \geq 0$ *such that the following condition is valid*

$$\left\| \begin{pmatrix} A_p^{-s} & 0 \\ 0 & A_p^{-s} \end{pmatrix} \begin{pmatrix} I - \tilde{B}_p^{1,2}(A_p) & \hat{B}_p^{1,2}(A_p) \\ \tilde{B}_p^{3,4}(A_p) & I - \hat{B}_p^{3,4}(A_p) \end{pmatrix}^{-1} \begin{pmatrix} A_p^{-1} & 0 \\ 0 & A_p^{-1} \end{pmatrix} \right\|_{\mathbf{R}^{2p\times 2p}} < \beta, \ \forall p \in \mathbf{N},$$

then the nonlocal in time problem (2), (3) has a unique solution and the mapping $\{u_0, u_1, f\} \to \{u, u'\}$ *is linear and continuous from the space* $V \times H \times L^2(0,T;H)$ *to the space* $C^0([0,T];V) \times C^0([0,T];H)$.

Note that the existence and uniqueness result for the nonclassical problem (2), (3) is also valid in the case of noncompact nonlocal operators B^1, B^2, B^3 and B^4.

Theorem 2. *Let the linear functionals* b_{jk}^1, b_{jk}^2, b_{jk}^3 *and* b_{jk}^4 *defined by the operators* $B^1 \in L(C^0([0,T];V);V)$, $B^2 \in L(C^0([0,T];H);V)$, $B^3 \in L(C^0([0,T];V);H)$ *and* $B^4 \in L(C^0([0,T];H);H)$ *satisfy the following conditions* $b_{jk}^1 \equiv b_{jk}^2 \equiv b_{jk}^3 \equiv b_{jk}^4 \equiv 0$, *for all* $j < k$, $k \in \mathbf{N}$, $k \geq 2$, *and there exists a constant* $\beta = const > 0$ *such that*

$$\left\| \begin{pmatrix} A_p & 0 \\ 0 & A_p \end{pmatrix} \begin{pmatrix} I - \tilde{B}_p^{1,2}(A_p) & \hat{B}_p^{1,2}(A_p) \\ \tilde{B}_p^{3,4}(A_p) & I - \hat{B}_p^{3,4}(A_p) \end{pmatrix}^{-1} \begin{pmatrix} A_p^{-1} & 0 \\ 0 & A_p^{-1} \end{pmatrix} \right\|_{\mathbf{R}^{2p\times 2p}} < \beta, \ \forall p \in \mathbf{N}.$$

If $u_0 \in V$, $u_1 \in H$ *and* $f \in L^2(0,T;H)$, *then the nonclassical problem* (2), (3) *has a unique solution and the following estimate is valid*

$$\| u \|_{C^0([0,T];V)} + \| u' \|_{C^0([0,T];H)} \leq c(\| u_0 \|_V + \| u_1 \|_H + \| f \|_{L^2(0,T;H)}). \tag{4}$$

For nonlocal in time problem (2), (3) we can construct an iteration algorithm of approximation in corresponding spaces of solution of the nonclassical problem by solutions of classical problems. We consider the following sequence of problems with classical initial conditions: Find a vector-function $w_p \in C^0([0,T];V)$, $w_p' \in C^0([0,T];H)$, which satisfies the following equation in the sense of distribution on $(0,T)$ and in suitable spaces the initial conditions

$$\frac{d}{dt}(w_p'(.), v)_H + a(w_p(.), v) = (f(.), v)_H, \ \forall v \in V,$$
$$w_p(0) = B^1 w_{p-1} + B^2 w_{p-1}' + u_0, \ w_p'(0) = B^3 w_{p-1} + B^4 w_{p-1}' + u_1, \ p \geq 1, \tag{5}$$

where $w_0 \equiv 0$. For arbitrary operators $B^1 \in L(C^0([0,T];V);V)$, $B^2 \in L(C^0([0,T];H);V)$, $B^3 \in L(C^0([0,T];V);H)$, $B^4 \in L(C^0([0,T];H);H)$ and vectors $u_0 \in V$, $u_1 \in H$, we have $w_p(0) \in V$, $w_p'(0) \in H$ and by Theorem 2, if $f \in L^2(0,T;H)$, then the classical problem (5) has a unique solution for each $p \in \mathbf{N}$. The following theorem gives conditions on the nonlocal operators B^1, B^2, B^3 and B^4 for which the sequence of vector-functions $(w_p)_{p=1}^{\infty}$ converges to the solution of the nonclassical problem.

Theorem 3. *Suppose that the operators* B^1, B^2, B^3 *and* B^4 *satisfy the following conditions* $b_{jk}^1 \equiv b_{jk}^2 \equiv b_{jk}^3 \equiv b_{jk}^4 \equiv 0$, *for all* $j < k$, $k \in \mathbf{N}$, $k \geq 2$, *and there exists constant* $0 < \beta < 1$ *such that*

$$\left\| \begin{pmatrix} A_p & 0 \\ 0 & A_p \end{pmatrix} \begin{pmatrix} \tilde{B}_p^{1,2}(A_p) & \hat{B}_p^{1,2}(A_p) \\ \tilde{B}_q^{3,4}(A_p) & \hat{B}_p^{3,4}(A_p) \end{pmatrix} \begin{pmatrix} A_p^{-1} & 0 \\ 0 & A_p^{-1} \end{pmatrix} \right\|_{\mathbf{R}^{2p\times 2p}} < \beta, \ \forall p \in \mathbf{N}.$$

If $u_0 \in V$, $u_1 \in H$ *and* $f \in L^2(0,T;H)$, *then the nonlocal problem* (2), (3) *has a unique solution u and the sequence* $(w_p)_{p=1}^{\infty}$ *of solutions of problems* (5) *tends to u in the space* $C^0([0,T];V)$, *and* $(w_p')_{p=1}^{\infty}$ *tends to* u' *in the space* $C^0([0,T];H)$.

It should be pointed out that the approximation algorithm (5) of the solution of the nonlocal in time problem by solutions of classical problems permits one to investigate the nonclassical problem (2), (3) if the operator A satisfies only self-adjointness, continuity and strong coerciveness conditions (1), and the spectrum of the operator A does not satisfy any additional conditions required in previous theorems, which hold if the embedding of the space V in H is compact [9].

Theorem 4. *If $u_0 \in V$, $u_1 \in H$, $f \in L^2(0,T;H)$, the operator $A \in L(V;V')$ satisfies conditions* (1) *and the norms of the operators $B^1 \in L(C^0([0,T];V);V)$, $B^2 \in L(C^0([0,T];H);V)$, $B^3 \in L(C^0([0,T];V);H)$ and $B^4 \in L(C^0([0,T];H);H)$ are such that*

$$c_a\left(1+\varepsilon_1\right)\| B^1 \|_L^2 + \left(1+\frac{1}{\varepsilon_2}\right)\| B^3 \|_L^2 < \chi \hat{c}_a, \quad c_a\left(1+\frac{1}{\varepsilon_1}\right)\| B^2 \|_L^2 + \left(1+\varepsilon_2\right)\| B^4 \|_L^2 < 1-\chi,$$

for some $0<\chi<1$, $0<\varepsilon_1<1$, $0<\varepsilon_2<1$, then the nonclassical problem (2), (3) *has a unique solution and the estimate* (4) *is valid.*

Now let us consider some applications of the general results obtained for abstract nonclassical problems to nonlocal in time problems for hyperbolic partial differential equations and systems. Let $\Omega \subset \mathbf{R}^n$, $n \in \mathbf{N}$, be a bounded domain with boundary Γ of the class $C^{r-1,1}$, $r \in \mathbf{N}$ [9]. By $H^k(\Omega)$ we denote the Sobolev space of order k based on $L^2(\Omega)$. We denote the closure in $H^k(\Omega)$ of the set $D(\Omega)$ of infinitely differentiable functions with compact support in Ω by $H_0^k(\Omega)$ and its dual space we denote by $H^{-k}(\Omega)$. We set $\mathbf{H}_0^k(\Omega) = [H_0^k(\Omega)]^N$, $\mathbf{L}^2(\Omega) = [L^2(\Omega)]^N$ and consider the $2r$th-order elliptic operator

$$A \equiv \sum_{j=0}^{r}(-1)^j \sum \frac{\partial^j}{\partial x_{n_1}\ldots\partial x_{n_j}}\left(A_{q_1\ldots q_j}^{n_1\ldots n_j}(x)\frac{\partial^j}{\partial x_{q_1}\ldots\partial x_{q_j}}\right),$$

where the summation in the inner sum is performed over all possible sets of indices $n_1,\ldots,n_j$, $q_1,\ldots,q_j$ each of which independently ranges over the values $\{1,\ldots,p\}$, $A_{q_1\ldots q_j}^{n_1\ldots n_j}(x)$ is a square matrix of order N that is not changed under any permutation of subscripts or superscripts, and the replacement of the superscripts by the subscripts gives the adjoint matrix. We assume that the entries of the matrices belong to $L^\infty(\Omega)$ and the inequalities

$$\begin{aligned}&\left(\sum A_{q_1\ldots q_r}^{n_1\ldots n_r}(x)\vec{b}_{q_1\ldots q_r},\vec{b}_{n_1\ldots n_r}\right)_{\mathbf{R}^N} \ge c\sum \| \vec{b}_{q_1\ldots q_r} \|_{\mathbf{R}^N}^2, \quad c = const > 0,\\ &\left(\sum A_{q_1\ldots q_j}^{n_1\ldots n_j}(x)\vec{b}_{q_1\ldots q_j},\vec{b}_{n_1\ldots n_j}\right)_{\mathbf{R}^N} \ge 0, \quad j = 0,1,\ldots,r-1,\end{aligned} \tag{6}$$

are valid for almost all $x \in \Omega$, $\vec{b}_{q_1\ldots q_j}$ is an arbitrary N-component vector that is preserved under a permutation of $q_1,\ldots,q_j$ $(j=\overline{1,r})$. Partial derivatives in the definition of A are treated as generalized derivatives with respect to the corresponding variables. Note that the jth-order $(j=1,\ldots,r)$ partial derivative of a function in $L^2(\Omega)$ belongs to the space $H^{-j}(\Omega)$ $(j=\overline{1,r})$. Since the multiplication of a function in the space $L^2(\Omega)$ by a function in $L^\infty(\Omega)$ leaves it in $L^2(\Omega)$, for each vector-function $v \in \mathbf{H}_0^r(\Omega) = (H_0^r(\Omega))^N$ we have $Av \in \mathbf{H}^{-r}(\Omega) = (H^{-r}(\Omega))^N$. The elliptic operator A is a continuous operator from $\mathbf{H}_0^r(\Omega)$ to the space $\mathbf{H}^{-r}(\Omega)$, which, by virtue of (6), satisfies the following self-adjointness and strong coerciveness conditions

$$\langle Av,w\rangle_r = \langle Aw,v\rangle_r, \quad \langle Av,v\rangle_r \ge \alpha \| v \|_{\mathbf{H}^r(\Omega)}^2, \quad \forall v,w \in \mathbf{H}_0^r(\Omega),$$

where $\alpha = const > 0$, $\langle .,. \rangle_r$ is the duality relation between the spaces $\mathbf{H}^{-r}(\Omega)$ and $\mathbf{H}_0^r(\Omega)$. The bilinear form $a(v,w) = \langle Av,w\rangle_r$ corresponding to the operator A is of the following form

$$a(v,w)=\int_{\Omega}\sum_{j=0}^{r}\sum\left(A_{q_1\dots q_j}^{n_1\dots n_j}(x)\frac{\partial^j v}{\partial x_{q_1}\dots\partial x_{q_j}},\frac{\partial^j w}{\partial x_{n_1}\dots\partial x_{n_j}}\right)_{\mathbf{R}^N}dx$$

and satisfies conditions (1). Since the continuous embedding of $\mathbf{H}_0^r(\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact, it follows from properties of self-adjoint coercive operators mapping a Hilbert space into its dual [9] that there exists a system of orthonormal in $\mathbf{L}^2(\Omega)$ and complete in $\mathbf{H}_0^r(\Omega)$ eigenfunctions $\{v_n\}_{n=1}^{\infty}$ of the operator $A:\mathbf{H}_0^r(\Omega)\to\mathbf{H}^{-r}(\Omega)$ corresponding to the eigenvalues $\{|\lambda_n|^2\}_{n=1}^{\infty}$, such that $0<\lambda_1\le\lambda_2\le\dots$, $\lambda_n\to\infty$ as $n\to\infty$.

Let us consider the nonclassical problem for hyperbolic system of partial differential equations

$$\frac{\partial^2 u}{\partial t^2}+Au=f(x,t),\quad (x,t)\in\Omega\times(0,T),\tag{7}$$

with the nonlocal initial and homogeneous boundary conditions

$$\begin{aligned}&u(x,0)=\sum_{j=1}^{m}\alpha_j^1 u(x,T_j)+\int_0^T\rho_1(\tau)u(x,\tau)d\tau+u_0(x),\\&\frac{\partial u}{\partial t}(x,0)=\sum_{j=1}^{m}\alpha_j^2 u(x,T_j)+\int_0^T\rho_2(\tau)u(x,\tau)d\tau+\sum_{j=1}^{m}\alpha_j^3\frac{\partial u}{\partial t}(x,T_j)+\int_0^T\rho_3(\tau)\frac{\partial u}{\partial t}(x,\tau)d\tau+u_1(x),\end{aligned}\quad x\in\Omega,\tag{8}$$

$$u(x,t)=\frac{\partial u}{\partial \mathbf{v}}(x,t)=\dots=\frac{\partial^{r-1}u}{\partial\mathbf{v}^{r-1}}(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T),\tag{9}$$

where $\mathbf{v}$ is the unit normal to the boundary $\partial\Omega$, $\alpha_j^1,\alpha_j^2,\alpha_j^3\in\mathbf{R}$ $0<T_j\le T$ $(j=\overline{1,m})$ and $\rho_1,\rho_2,\rho_3\in L^1(0,T)$. Note that vector-functions defined on $\Omega\times(0,T)$ can be identified with vector-functions defined on $(0,T)$ and ranging in the corresponding function spaces on Ω. Hence, the nonlocal in time problem (7)-(9) can be stated as the problem of finding a vector-function $u\in C^0([0,T];\mathbf{H}_0^r(\Omega))$, which satisfies equation (7) in the space $D'(0,T;\mathbf{H}^{-r}(\Omega))$ of distributions on $(0,T)$ ranging in $\mathbf{H}^{-r}(\Omega)$, where the partial derivative $\partial^2u/\partial t^2$ is treated as second-order generalized derivative $u''=d^2u/dt^2\in D'(0,T;\mathbf{H}_0^r(\Omega))$. In addition, we seek for $u\in C^0([0,T];\mathbf{H}_0^r(\Omega))$ such that $u'\in C^0([0,T];L^2(\Omega))$, u satisfies conditions (8) in the spaces $\mathbf{H}_0^r(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively, and boundary conditions (9) are valid, since for each vector-function in $\mathbf{H}_0^r(\Omega)$, the traces of its generalized derivatives of order not exceeding r–1 vanish on the boundary of the domain Ω. Thus, the nonlocal in time problem (7)-(9) is a particular case of the nonclassical problem (2), (3), with $V=\mathbf{H}_0^r(\Omega)$ and $H=\mathbf{L}^2(\Omega)$. To formulate the corresponding theorem on the existence and uniqueness of solution of the nonlocal in time problem (7)-(9) we use the following notations

$$\begin{aligned}\zeta_{i,q}&=\sum_{j=1}^{m}\alpha_j^i\cos(\lambda_q T_j)+\int_0^T\rho_i(\tau)\cos(\lambda_q\tau)d\tau,\\ \xi_{i,q}&=\sum_{j=1}^{m}\alpha_j^i\sin(\lambda_q T_j)+\int_0^T\rho_i(\tau)\sin(\lambda_q\tau)d\tau,\end{aligned}\qquad i=1,2,3,\quad q\in\mathbf{N}.$$

Theorem 5. *If* $u_0\in\mathbf{H}_0^r(\Omega)$, $u_1\in\mathbf{L}^2(\Omega)$, $f\in L^2(0,T;\mathbf{L}^2(\Omega))$ *and there exists a constant* $\gamma>0$ *such that*

$$\left|(1-\zeta_{1,q})(1-\zeta_{3,q}-\frac{1}{\lambda_q}\xi_{2,q})+\xi_{1,q}(\xi_{3,q}-\frac{1}{\lambda_q}\zeta_{2,q})\right|>\gamma,\quad\forall q\in\mathbf{N},\tag{10}$$

then the nonclassical problem (7)-(9) has a unique solution $u\in C^0([0,T];\mathbf{H}_0^r(\Omega))$, $u'\in C^0([0,T];\mathbf{L}^2(\Omega))$ *and the following estimate is valid*

$$\|u\|_{C^0([0,T];\mathbf{H}_0^r(\Omega))}+\|u'\|_{C^0([0,T];\mathbf{L}^2(\Omega))}\le c(\|u_0\|_{\mathbf{H}_0^r(\Omega)}+\|u_1\|_{\mathbf{L}^2(\Omega)}+\|f\|_{L^2(0,T;\mathbf{L}^2(\Omega))}).$$

Moreover, solution of the nonlocal in time problem (7)-(9) is unique if and only if condition (10) is valid for $\gamma\ge 0$.

Applying the latter theorem we can obtain sufficient conditions on the coefficients $\alpha_j^1,\alpha_j^2,\alpha_j^3$ $(j=\overline{1,m})$ and weight functions ρ_1,ρ_2,ρ_3, for which there exists a unique solution of the nonclassical initial-boundary value problem (7)-(9).

Corollary 1. *If* $u_0\in\mathbf{H}_0^r(\Omega)$, $u_1\in\mathbf{L}^2(\Omega)$, $f\in L^2(0,T;\mathbf{L}^2(\Omega))$ *and* $\alpha_j^1,\alpha_j^2,\alpha_j^3$ $(j=\overline{1,m})$, ρ_1,ρ_2,ρ_3 *satisfy the condition*

$$\sum_{j=1}^m(|\alpha_j^1|+\frac{1}{\lambda_1}|\alpha_j^2|+|\alpha_j^3|)<1-\int_0^T|\rho_1(\tau)|\,d\tau-\frac{1}{\lambda_1}\int_0^T|\rho_2(\tau)|\,d\tau-\int_0^T|\rho_3(\tau)|\,d\tau,$$

then the nonlocal in time problem (7)-(9) has a unique solution.

Corollary 2. *If* $u_0\in\mathbf{H}_0^r(\Omega)$, $u_1\in\mathbf{L}^2(\Omega)$, $f\in L^2(0,T;\mathbf{L}^2(\Omega))$ $\alpha_j^1=\alpha_j^3=\alpha_j$ $(j=\overline{1,m})$, $\rho_1\equiv\rho_3\equiv\rho$ *and satisfy the condition*

$$\sum_{j=1}^m|\alpha_j|<1-\int_0^T|\rho(\tau)|\,d\tau-\sqrt{\frac{2}{\lambda_1}\left(\sum_{j=1}^m|\alpha_j|+\int_0^T|\rho(\tau)|\,d\tau\right)\left(\sum_{j=1}^m|\alpha_j^2|+\int_0^T|\rho_2(\tau)|\,d\tau\right)},$$

then the nonlocal in time problem (7)-(9) has a unique solution.

Now let us consider the nonclassical problem (7)-(9) with $\alpha_j^1=\alpha_j^3=\alpha_j$, $\alpha_j^2=0$ $(j=\overline{1,m})$, and $\rho_1\equiv\rho_2\equiv\rho_3\equiv 0$. By Corollary 2, if $\sum_{j=1}^m|\alpha_j|<1$, then the nonlocal problem has a unique solution for all $u_0\in\mathbf{H}_0^r(\Omega)$, $u_1\in\mathbf{L}^2(\Omega)$, $f\in L^2(0,T;\mathbf{L}^2(\Omega))$. If $\sum_{j=1}^m|\alpha_j|=1$, then it follows from Theorem 5 that the nonlocal in time problem has at most one solution if and only if $|\lambda_q|T_j\ne\pi(2k-1-\chi(\alpha_j))$ for at least one value of j $(j=1,...,m)$ and for all $q,k\in\mathbf{N}$, where $\chi(.)$ is Heaviside function, $\chi(0)=0$. For these coefficients α_j the following existence theorem is valid, where by $\mathrm{Int}(y)$ we denote the integer part of a real number y.

Theorem 6. *Let* $\alpha_j^1=\alpha_j^3=\alpha_j$, $\alpha_j^2=0$ $(j=\overline{1,m})$, $\rho_1\equiv\rho_2\equiv\rho_3\equiv 0$, $\sum_{j=1}^m|\alpha_j|=1$ *and for some* $j\in J\subset\{1,...,m\}$ *there exist positive constants* $\hat{c}_j>0$ *and nonnegative integers* $\hat{s}_j\in\mathbf{N}\cup\{0\}$ *such that*

$$\left||\lambda_q|T_j-\pi(2k-1-\chi(\alpha_j))\right|\ge\frac{\hat{c}_j}{|\lambda_q|^{\hat{s}_j}},\quad\forall k,q\in\mathbf{N}. \tag{11}$$

If $u_0\in\mathbf{H}_0^r(\Omega)\cap\mathbf{H}^{(\hat{s}+1)r}(\Omega)$, $u_1\in\mathbf{H}^{\hat{s}r}(\Omega)$, $f\in L^2(0,T;\mathbf{H}^{\hat{s}r}(\Omega))$, $\hat{s}=\hat{s}_{j_0}\min\{2,m\}$, $\hat{s}_{j_0}=\min_{j\in J}\{\hat{s}_j\}$, $Au_0,\ldots,$ $A^{\mathrm{Int}(\hat{s}/2)}u_0\in\mathbf{H}_0^r(\Omega)$ *and* u_1, $Au_1,\ldots,A^{\mathrm{Int}((\hat{s}-1)/2)}u_1\in\mathbf{H}_0^r(\Omega)$, *f*, *Af...*, $A^{\mathrm{Int}((\hat{s}-1)/2)}f\in L^2(0,T;\mathbf{H}_0^r(\Omega))$ *for* $\hat{s}\ge 1$, *then the nonlocal in time problem (7)-(9) has a unique solution and the following estimate is valid*

$$\|u\|_{C^0([0,T];\mathbf{H}_0^r(\Omega))}+\|u'\|_{C^0([0,T];\mathbf{L}^2(\Omega))}\le c(\|u_0\|_{\mathbf{H}^{(\hat{s}+1)r}(\Omega)}+\|u_1\|_{\mathbf{H}^{\hat{s}r}(\Omega)}+\|f\|_{L^2(0,T;\mathbf{H}^{\hat{s}r}(\Omega))}).$$

Using the latter theorem we investigate the dependence of the existence and uniqueness of solution of nonlocal in time problem on algebraic properties of numbers characterizing the geometric shape of the spatial domain and values of time variable in the nonclassical initial conditions. Let us consider the nonclassical problem for one-dimensional wave equation

$$\frac{\partial^2u}{\partial t^2}-\frac{\partial^2u}{\partial x^2}=f(x,t),\ (x,t)\in(0,l)\times(0,T), \tag{12}$$

with homogeneous boundary and nonlocal initial conditions

$$u(0,t) = u(l,t) = 0, \ \ t \in (0,T), \tag{13}$$

$$\begin{aligned} u(x,0) &= \sum_{j=1}^{m} \alpha_j u(x,T_j) + u_0(x), \\ \frac{\partial u}{\partial t}(x,0) &= \sum_{j=1}^{m} \alpha_j \frac{\partial u}{\partial t}(x,T_j) + u_1(x), \end{aligned} \qquad x \in (0,l), \tag{14}$$

where $\alpha_j \neq 0$ are given real numbers, $0 < T_j \le T \ \ (j = 1,...,m)$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$, and u is the unknown function in the space $C^0([0,T];H_0^1(\Omega))$ and $u' \in C^0([0,T];L^2(\Omega))$. If $\sum_{j=1}^{m} |\alpha_j| < 1$, then the nonlocal in time problem has a unique solution, and in the case of $\sum_{j=1}^{m} |\alpha_j| = 1$, the problem has at most one solution if and only if at least one of the points $T_j \ \ (1 \le j \le m)$ is such that the ratio T_j/l is irrational or can be represented as a ratio of even and odd numbers, when $\alpha_j < 0$. Note that the condition of irrationality of the numbers T_j/l is insufficient for the existence of a solution even for analytic given functions u_0, u_1, f. If $\alpha_j < 0$ and $T_j/l = 2n/(2n_1 + 1)$, for some $n, n_1 \in \mathbf{N}$, then T_j/l satisfies condition (11) with $\hat{s}_j = 0$ and, consequently, the nonlocal in time problem has a unique solution. So, in the case of positive coefficients α_j for the nonlocal in time problem to be solvable, it is necessary to impose constraints on the order of approximation of at least one irrational $T_j/l \ \ (j = \overline{1,m})$ by rational numbers. Theorem 6 with $\min_{j \in J}\{\hat{s}_j\} = 1,2$ implies the following results on the existence of a solution to the nonclassical problem (12)-(14).

Theorem 7. *If* $\sum_{j=1}^{m} |\alpha_j| = 1$, *for at least one* $T_{j_0}/l \ \ (1 \le j_0 \le m)$ *there exists a positive constant c>0 such that*

$$\left| \frac{T_{j_0}}{l} - \frac{k}{q} \right| \ge \frac{c}{q^2}, \ \ \forall k,q \in \mathbf{N}, \tag{15}$$

and $u_0 \in H^2(0,l) \cap H_0^1(0,l)$, $u_1 \in H_0^1(0,l)$, $f \in L^2(0,T;H_0^1(0,l))$ *for* $m=1$, *or* $u_0 \in H^3(0,l) \cap H_0^1(0,l)$, $u_1 \in H^2(0,l) \cap H_0^1(0,l)$, $f \in L^2(0,T;H^2(0,l) \cap H_0^1(0,l))$, $\partial^2 u_0/\partial x^2 \in H_0^1(0,l)$, *for* $m \ge 2$, *then the nonlocal in time problem (12)-(14) has a unique solution.*

Theorem 8. *If* $\sum_{j=1}^{m} |\alpha_j| = 1$, *for at least one* $T_{j_0}/l \ \ (1 \le j_0 \le m)$ *there exists* $0 < \varepsilon < 1$ *and corresponding positive constant* $c(\varepsilon) > 0$ *such that*

$$\left| \frac{T_{j_0}}{l} - \frac{k}{q} \right| \ge \frac{c(\varepsilon)}{q^{2+\varepsilon}}, \ \ \forall k,q \in \mathbf{N}, \tag{16}$$

and $u_0 \in H^3(0,l) \cap H_0^1(0,l)$, $u_1 \in H^2(0,l) \cap H_0^1(0,l)$, $f \in L^2(0,T;H^2(0,l) \cap H_0^1(0,l))$, $\partial^2 u_0/\partial x^2 \in H_0^1(0,l)$ *for* $m = 1$, *or* $u_0 \in H^5(0,l) \cap H_0^1(0,l)$, $\partial^2 u_0/\partial x^2$, $\partial^4 u_0/\partial x^4 \in H_0^1(0,l)$, $u_1 \in H^4(0,l) \cap H_0^1(0,l)$, $\partial^2 u_1/\partial x^2 \in H_0^1(0,l)$, $f \in L^2(0,T;H^4(0,l) \cap H_0^1(0,l))$, $\partial^2 f/\partial x^2 \in L^2(0,T;H_0^1(0,l))$, *for* $m \ge 2$, *then the nonlocal in time problem* (12)-(14) *has a unique solution.*

Note that the set of numbers T_{j_0}/l satisfying condition (15) contains all irrational algebraic numbers of degree 2, and, by the generalization of the Liouville theorem on the approximation of irrational numbers by rational ones [10], each irrational algebraic number T_{j_0}/l satisfies inequality (16).

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არაკლასიკური ამოცანები არალოკალური საწყისი პირობებით აბსტრაქტული მეორე რიგის ევოლუციური განტოლებებისათვის

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REFERENCES

1. *M. Ehrendorfer* (1992), Quart. J. Roy. Meteorol. Sci., **118**, 566: 673-713.
2. *V.V. Shelukhin* (1993), Dinamika sploshnoi sredy, **107**: 180-193 (in Russian).
3. *V.V. Shelukhin* (1995), Sibirsk. Mat. Zh., **36**: 701-724 (in Russian).
4. *A.A. Kerefov* (1979), Diff. Uravn., **15**: 74-78 (in Russian).
5. *D. Gordeziani* (1989), Rep. of Enlarged Sess. of I. Vekua Inst. Appl. Math., **4**: 57-60.
6. *C.V. Pao* (1995), J. Math. Anal. Appl., **195**: 702-718.
7. *D. Gordeziani, M. Avalishvili, G. Avalishvili* (2002), Appl. Math. and Inform., **7**: 66-77.
8. *R. Dautray, J.-L. Lions* (1988), Analyse mathématique et calcul numérique pour les sciences et les techniques, vol. 8, Paris.
9. *W. McLean* (2000), Strongly elliptic systems and boundary integral equations. Cambridge University Press, Cambridge.
10. *K.F. Roth* (1955), Mathematika, **2**: 1-20.

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