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Mathematics

A Distribution Maximum Inequality for Rearrangements of Summands

Sergei Chobanyan*, Shlomo Levental, Habib Salehi****

*Niko Muskhelishvili Institute of Computational Mathematics of Georgian Technical University; International School of Economics of I. Javakhishvili Tbilisi State University, Tbilisi
**Department of Statistics and Probability of Michigan State University, East Lansing, USA

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ABSTRACT. **We state the following maximum inequality on rearrangement of summands. Let** $x_1, \ldots, x_n$, $\sum_1^n x_i = 0$ **be a collection of elements of a normed space** ***X*****. Then for any collection of signs** $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ **and any** $t > 0$

$$\text{card}\,\{\pi : \max_{1\le k\le n} \| \sum_1^k x_{\pi(i)} \| > t\} \le C \quad \text{card}\,\{\pi : \max_{1\le k\le n} \| \sum_1^k x_{\pi(i)} \vartheta_i \| > \frac{t}{C}\},$$

where $\pi \in \Pi_n$, Π_n **is the group of all permutations of** $\{1, \ldots, n\}$ **and** $C > 0$ **is an absolute constant. The inequality is unimprovable (the inverse inequality also holds for some other constant) and generalizes well-known results due to Garsia, Maurey and Pisier, Kashin and Saakyan, Chobanyan and Salehi, and Levental.** *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *permutations of summands, maximum inequality for rearrangements of summand.*

1. Introduction

The main purposes of this paper is to study the distribution of the random variable $| x_\pi |_n = \max_{1\le k\le n} \| x_{\pi(1)} + \ldots + x_{\pi(k)} \|$ where $x = (x_1, \ldots, x_n) \in X$, X is a normed space that is defined on Π_n, the set of all permutations of $\{1, \ldots, n\}$ with the uniform probability on it. There are a series of problems and results in analysis where this sort of rearrangement maximum inequalities are used, see e.g. the following sources and the literature therein: The Levy-Steinitz theorem on the sum range of a conditionally convergent series (M.I.Kadets and V.M.Kadets [1]), Nikishin type theorems on a.s. convergence of rearranged functional series (Levental et al. [2]), orthogonal series (Kashin and Saakyan [3]), Kolmogorov conjecture on systems of convergence (Bourgain [4]), the Ulyanov problem on the uniform convergence of a rearrangement of the trigonometric Fourier series of a periodic continuous function (Konyagin [5] and Sz.Gy.Revesz [6]) and the applications of compact vector summation in scheduling theory (Sevastyanov [7]).

The first result in this direction was found by M.Kadets [8] who was solving the Steinitz problem for the L_p-spaces.

Garsia in [9] and [10] proved that in the 1-dimensional case $(X = R^1)$ for any $x=(x_1,..,x_n) \in X^n$ with $\sum_1^n x_i = 0$,

$$E_\pi |x_\pi|_n^p \le C(\sum_1^n |x_i|^2)^{p/2},$$

where E_π is the average (expectation), $p \ge 1$ and $C > 0$ is a constant dependent on p only. This inequality led to the well-known Garsia theorem on a.s. convergent rearrangement of an orthogonal series.

Maurey and Pisier [11] were first to show the relationship between the permutations and signs. They proved that in the general case of a normed space X for any $x=(x_1,..,x_n) \in X$ with $\sum_1^n x_i = 0$, and for any $p \ge 1$

$$E_\pi |x_\pi|_n^p \sim E_r \| \sum_1^n x_i r_i \|^p, \tag{1}$$

where $r_1,...,r_n$ are Rademacher random variables. The relation of equivalence in (1) means that the ratios are bounded by positive constants. It is amazing that for a long time (until the early 90s), the result of Maurey and Pisier remained unknown. Meanwhile Kashin and Saakyan [3] have proved the following result for $X = R^1$ in terms of distributions: there exists a universal constant C>0 such that for any t>0 and any reals $x_1,..,x_n$ with $\sum_1^n x_i = 0$ the right-hand-side fragment of the following inequality holds

$$P_r\{\omega : \| \sum_1^n x_i r_i(\omega)\| > 2t\} \le P_\pi\{\pi : |x_\pi|_n > t\} \le C P_r\{\omega : \| \sum_1^n x_i r_i(\omega)\| > \frac{t}{C}\}. \tag{2}$$

However, the method used in [3] does not work in the case of vectors. Then Chobanyan [12] and Chobanyan and Salehi [13] used a different method based on Lemma 1 below to prove the two-sided inequality (2) for a general normed space X. In Levental [14] for the case $x_i = \pm 1$ and in Levental [15] for the case of $x_i \in R^1$, $i = 1,...,n$ the inequality (2) was given the following form. *There are universal constants* C_1 *and* C_2 *such that for any reals* $x_1,..,x_n$ *with* $\sum_1^n x_i = 0$, *any collection of signs with* $\vartheta = (\vartheta_1,...,\vartheta_n)$ *and any* t>0 *the following inequality holds:*

$$C_1 P_\pi\{\pi : |x_\pi \vartheta|_n > \frac{t}{C_1}\} \le P_\pi\{\pi : |x_\pi \vartheta|_n > t\} \le C_2 P_\pi\{\pi : |x_\pi \vartheta|_n > \frac{t}{C_2}\}, \tag{3}$$

where

$$|x_\pi \vartheta|_n = \max_{1 \le k \le n} |x_{\pi(1)}\vartheta_1 + ... + x_{\pi(k)}\vartheta_k|.$$

Inequality (3) can be regarded as a refinement of (2) for a general normed space X. As a corollary we single out the following curious result: *There exist universal constants* C_1 *and* C_2 *such that for any finite collection* $x=(x_1,..,x_n)$ *of elements of a normed space X with* $\sum_1^n x_i = 0$, *any collection of signs* $\vartheta = (\vartheta_1,...,\vartheta_n)$ *and any* t>0 *the following inequality holds for the distribution of the Rademacher sum:*

$$C_1 P\{\pi : |x_\pi \vartheta|_n > \frac{t}{C_1}\} \le P\{\omega : \| \sum_1^n x_i r_i(\omega)\| \ge t\} \le C_2 P\{\pi : |x_\pi \vartheta|_n > \frac{t}{C_2}\}.$$

Let us note that inequality (3) (as well as the majority of inequalities of this paper) can be expressed in terms of exchangeable random variables: *There are universal constants* C_1 *and* C_2 *such that for any finite exchangeable system* $\xi = (\xi_1,..,\xi_n)$ *of random variables with* $\sum_1^n \xi_i = 0$, *any collection of signs* $\vartheta = (\vartheta_1,...,\vartheta_n)$ *and any* t>0 *the following inequality holds:*

$$C_1 P\{\omega: |\xi(\omega)\ \vartheta|_n > \frac{t}{C_1}\} \le P\{\omega: |\xi(\omega)|_n > t\} \le C_2 P\{\omega: |\xi(\omega)\vartheta|_n > \frac{t}{C_2}\}. \quad (3')$$

In Section 2 we give a lemma (Lemma 1) first proved in [12,13,14] which simplifies drastically the proofs of moment maximum inequalities for convex increasing functions (Section 3). In Section 4 we state the main theorem on two-sided inequalities for the tail probabilities that lead via the integration by parts formula to moment inequalities for arbitrary increasing continuous functions.

2. Preliminaries

X stands for a normed space real or complex denotes a normed space, real or complex with the norm $\|\cdot\|$, Π_n for the group of all permutations $\pi:\{1,...,n\}\to\{1,...,n\}$ and Θ_n for all collections of signs $\vartheta=(\vartheta_1,...,\vartheta_n)$, $\vartheta_i=\pm1$, $i=1,...,n$.

Given $x=(x_1,...,x_n)\in X^n$, $\pi\in\Pi_n$ and $\vartheta=(\vartheta_1,...,\vartheta_n)\in\Theta_n$ we denote

$$|x\vartheta|_n = \max_{1\le k\le n}\|x_1\vartheta_1+...+x_k\ \vartheta_n\|.$$

In particular, $|x|_n=\max_{1\le k\le n}\|x_1+...+x_k\|$.

Let $\pi\in\Pi_n$ be a permutation with $\pi=(k_1,...,k_n)$ and let $\vartheta=(\vartheta_1,...,\vartheta_n)\in\Theta_n$ with

$$\vartheta_{u_1}=+1,\ \dots,\vartheta_{u_s}=+1;\ u_1<u_2<...<u_s\ ;$$
$$\vartheta_{v_1}=-1,\ \dots,\vartheta_{v_t}=-1;\ v_1<v_2<...<v_t\ ;\quad s+t=n\ .$$

Below we use the permutation $\pi^*(\pi,\vartheta)$ defined as follows: $\pi^*=(u_1,u_2,...,u_s,...,v_t,v_{t-1},...,v_1)$.

By $\pi_O\in\Pi_n$ we denote an *optimal* permutation , i.e. a permutation such that $|x_{\pi_O}|_n\le|x_\pi|_n$ for any $\pi\in\Pi_n$.

Below we use repeatedly the following lemma (see [12,13,14]).

Lemma 1. (i) *If* $(x_1,...,x_n)\in X^n$ *with* $\sum_1^n x_i=0$ *and* $\vartheta=(\vartheta_1,...,\vartheta_n)\in\Theta_n$, *then*

$$|x_\pi| + |x_\pi\vartheta|_n \ge 2|x_{\pi^*}|_n \ge |x_\pi\vartheta|_n\ .$$

(ii) *For any* $\vartheta=(\vartheta_1,...,\vartheta_n)\in\Theta_n$

$$|x_{\pi_O}| \le |x_{\pi_O}\vartheta|\ .$$

3. The moment maximum inequalities for exchangeable random variables

A finite collection $\xi=(\xi_1,...,\xi_n)$ of X-valued random variables is called *exchangeable*, if for each $\pi\in\Pi_n$ the rearranged collection $\xi=(\xi_{\pi(1)},...,\xi_{\pi(n)})$ has the same distribution in X^n as $(\xi_1,...,\xi_n)$.

Theorem 1. *Let* $\xi=(\xi_1,...,\xi_n)$ *be an exchangeable system of X-valued random variables with* $\sum_1^n\xi_i=0$ *, and let* $\Phi:[0,\infty)\to[0,\infty)$ *be an increasing convex function. Then :*

(i) *For any collection* $\vartheta=(\vartheta_1,...,\vartheta_n)\in\Theta_n$ *the following two-sided inequality holds:*

$$E\,\Phi(\frac{1}{2}|\xi\vartheta|_n)\le E\,\Phi(|\xi|_n)\le E\,\Phi(|\xi\vartheta|_n)\,.$$

(ii) $EE_r\Phi(\frac{1}{2}\|\sum_1^n\xi_i r_i\|)\ \le\ E\,\Phi(|\xi|_n)\le\ 2EE_r\Phi(\|\sum_1^n\xi_i r_i\|)$,

where $r_1,...,r_n$ *are Rademacher random variables.*

Proof. Since Φ is increasing and convex, using Lemma 1 we get

$$\Phi(|\xi\vartheta|_n \ \geq\ \Phi\big(2|\xi_{\pi^*}|_n - |\xi|_n\big) \geq\ 2\ \Phi(|\xi_{\pi^*}|_n) - \Phi(|\xi|_n).$$

Taking the expectation of both sides and using the fact that ξ is an exchangeable system, we come to the right-hand fragment of (i). The left-hand fragment of (i) also follows from Lemma 1:

$$E\Phi(\frac{1}{2}|\xi\vartheta|_n) \leq E\Phi(\frac{1}{2}(|\xi^+|_s + |\xi^-|_t)) \leq E\Phi(|\xi_\pi|_n),$$

where $\xi^+ = (\xi_{u_1},...,\xi_{u_s})$; $\xi^- = (\xi_{v_1},...,\xi_{v_t})$; $u_1 < ... < u_s$ are indices for which $\vartheta(u_1) = +1,...,\vartheta(u_s) = +1$; $v_1 < ... < v_t$ are indices for which $\vartheta(u_1) = -1,...,\vartheta(u_t) = -1$ Part (ii) follows from (i) by integrating with respect to the Rademacher random variables and using the Levy inequality.

Remark. The fact that for exchangeable X-valued random variables $\xi_1,...,\xi_n$ with $\sum_1^n \xi_i = 0$

$$c_p \leq \frac{E\max_{1\leq k\leq n}\|\sum_1^k \xi_i\|^p}{EE_r\|\sum_1^k \xi_i r_k\|^p} \leq C_p,$$

where $1 \leq p < \infty$, $c_p > 0$, $C_p > 0$ are constants, was found by Maurey and Pisier [11], using combinatorial arguments.

Let $x = (x_1,...,x_n) \in X^n$ and consider the following random variables on the probability space (Π_n, μ), where μ is the uniform distribution on Π_n:

$$\xi_k(\pi) = x_{\pi(k)},\ \ \pi \in \Pi_n,\ k = 1,...,n.$$

Obviously, $\xi = (\xi_1,...,\xi_n)$ is a system of exchangeable random variables and Theorem 1 can be restated in this particular case as follows.

Corollary 1. *Let* $\Phi:[0,\infty) \to [0,\infty)$ *be a convex increasing function,* $x = (x_1,...,x_n) \in X^n$ *be such that* $\sum_1^n x_i = 0$. *Then*

(i) *For any collection* $\vartheta = (\vartheta_1,...,\vartheta_n) \in \Theta_n$ *the following two-sided inequality holds:*

$$E_\pi\,\Phi(\frac{1}{2}|x_\pi\vartheta|_n) \leq E_\pi\,\Phi(|x_\pi|_n) \leq E_\pi\Phi(|x_\pi\vartheta|_n).$$

(ii) $E_r\,\Phi(\frac{1}{2}\|\sum_1^n x_i r_i\|) \leq\ E_\pi\,\Phi(|x_\pi|_n)\ \leq\ 2E_r\Phi(\|\sum_1^n x_i r_i\|)$.

In the light of Corollary 1 of interest is the estimation of $E_r\Phi(\|\sum_1^n x_i r_i\|)$. It can be expressed through the coefficients x_i, $i = 1,...,n$ when X is a Banach lattice not containing l_n^∞ uniformly (see [2] and the literature therein). In the next corollary we give only the simple and popular case of a Hilbert space containing the well-known results of Garsia.

Corollary 2. *Let H be a Hilbert space and* $x = (x_1,...,x_n) \in H^n$. *Then for any* p, $1 \leq p < \infty$, *the following maximum inequality holds:*

$$c_p\,(\sum_1^n\|x_i\|^2)^{p/2} \leq \frac{1}{n!}\sum_\pi |x_\pi|_n^p \leq C_p\,(\sum_1^n\|x_i\|^2)^{p/2}, \tag{4}$$

where c_p *and* C_p *are constants dependent on* p *only.*

The right-hand-side fragment of (4) was proved by Garsia [9,10] for the case of real x-s.

4. Maximum inequalities for the tail probabilities.

In conclusion we give without proof the following inequality that is suggested by Theorem 1 and Corollaries 1 and 2 to it.

Theorem 2. *Let* $\xi=(\xi_1,...,\xi_n)$ *be an exchangeable system of* X-valued random variables with $\sum_1^n \xi_i = 0$. Then for any $\vartheta=(\vartheta_1,...,\vartheta_n)\in\Theta_n$ *and any* $t>0$ *the following two-sided inequality holds:*

$$P(|\xi\vartheta|_n > 2t) \le P(|\xi|_n > t) \le 2P(|\xi\vartheta|_n > \frac{t}{22}).$$

We can also state Theorem 2 in the following equivalent form.

Theorem 2′. *Let* $x=(x_1,...,x_n)\in X^n$ be such that $\sum_1^n x_i = 0$. Then for any $\vartheta=(\vartheta_1,...,\vartheta_n)\in\Theta_n$ *and any* $t>0$ *the following two-sided inequality holds:*

$$P(\pi:|x_\pi\vartheta|_n > 2t) \le P(\pi:|x_\pi|_n > t) \le 2P(\pi:|x_\pi\vartheta|_n > \frac{t}{22}).$$

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განაწილების ტიპის მაქსიმალური უტოლობა შესაკრებთა გადანაცვლებებისათვის

ს. ჩობანიანი*, შ. ლევენტალი, ჰ. სალეჰი****

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(წარმოდგენილია აკადემიის წევრის ვ. კოკილაშვილის მიერ)

ნაშრომში ანონსირებულია შემდეგი დებულება. დავუშვათ $x_1,...,x_n$, $\sum_1^n x_i = 0$, არის X ნორმირებული სივრცის ელემენტთა ერთობლიობა. მაშინ ნებისმიერი $t>0$ და ნებისმიერი $\vartheta=(\vartheta_1,...,\vartheta_n)$ ნიშანთა ერთობლიობისათვის სამართლიანია შემდეგი უტოლობა

$$\text{card}\{\pi:\max_{1\le k\le n}\|\sum_1^k x_{\pi(i)}\|>t\}\le C\ \ \text{card}\{\pi:\max_{1\le k\le n}\|\sum_1^k x_{\pi(i)}\vartheta_i\|>\frac{t}{C}\},$$

სადაც $\pi\in\Pi_n,\Pi_n$ არის $\{1, \ldots, n\}$-ის ყველა გადანაცვლებათა ჯგუფი, ხოლო C არის აბსოლუტური მუდმივა.

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