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Mathematics

Semimartingale Backward Equations with Convex Generator

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(Presented by Academy Member Elizbar Nadaraya)

ABSTRACT. We are concerned with semimartingale backward stochastic differential equations (BSDE) with convex generators of quadratic growth. We prove the existence of a solution for such equation driven by a continuous martingale with unbounded characteristic. We introduce a suitable optimization problem and prove that the corresponding value process satisfies the above mentioned BSDE. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *backward equation, semimartingale, optimization problem.*

Let M be a d-dimensional square integrable martingale defined on a filtered probability space $(\Omega,\ F,\ \{F_t\}_{0\le t\le T},\ P)$ satisfying the usual conditions. Note that the quadratic variation $\langle M\rangle$ is a matrix with components $\langle M^i, M^j\rangle$ $i, j\in\{1, \dots, d\}$. Recalling that each component $d\langle M^i, M^j\rangle$ is absolutely continuous with respect to $dC=\sum_i d\langle M^i\rangle$ and there exists a predictable process m taking values in $R^{d\times d}$ such that $d\langle M\rangle$ can be written in the following form: $d\langle M\rangle_s = m_s m_s' dC_s$.

We consider a backward stochastic differential equation (BSDE) of the form:

$$\begin{cases} Y_t = Y_0 - \int_0^t f(s,\ Z_s)dC_s + \int_0^t Z_s dM_s + L_t \\ Y_T = \eta \end{cases} \tag{1}$$

where the generator $f:[0;T]\times\Omega\times R^d\to R$ is a measurable function, $f(\cdot,\cdot,z)$ is a predictable process for any $z=(z^1, \dots, z^d)$ and η is an F_T-measurable random variable. The couple $(f,\ \eta)$ is called parameters of equation (1).

Definition. A solution of equation (1) is a triple $(Y,\ Z,\ L)$ where $\{Y_t\}_{0\le t\le T}$ is a semimartingale; $\{Z_t\}_{0\le t\le T}$ is a R^d valued predictable process with $E\int_0^T |m_s Z_s|^2\, dC_s<\infty$; $\{L_t\}_{0\le t\le T}$ is a square integrable martingale orthogonal to M and the triple $(Y,\ Z,\ L)$ satisfies the equation (1).

Sometimes we shall say that the solution is only the first component of the triple $(Y,\ Z,\ L)$, bearing in mind that

$\int ZdM + L$ is the martingale part of Y.

Denote by $\varepsilon_{\cdot,t}\left(\int f'(u)dM\right)$ the unique solution of the following linear stochastic differential equation:

$$\begin{cases} dX_s = X_s f'(s,u_s)dM_s \\ X_t = 1 \end{cases} \qquad t \le s \le T$$

where f' is the d-dimensional vector of measurable version of subdifferential of the function f.

We say that a continuous R^d valued martingale M is from the class BMO if for any $i = 1, \dots, d$ $\quad \sup_\tau \left\| E\left[\langle M^i\rangle_T - \langle M^i\rangle_\tau \mid F_\tau\right]\right\|_\infty < \infty$ where the sup is taken over all stopping times $0 \le \tau \le T$.

Now we can formulate the main result of the article.

Theorem. *Suppose that the filtration* $\{F_t\}_{0\le t\le T}$ *is continuous and* $\{M_t\}_{0\le t\le T}$ *is a martingale from the class BMO. Let the parameters* (f, η) *of equation (1) satisfy the following conditions:*

1) $f(t, \omega, \cdot)$ *is a differentiable and convex function for any* (t, ω).

2) There exists a predictable, non-negative process α_t *and a constant* $\gamma \ge 0$, *such that* $\sup_\tau \left\| E\left[\int_\tau^T \alpha_s dC_s \mid F_\tau\right]\right\|_\infty < \infty$ *and for any* (t, ω, z)

$$|f(t, \omega, z)| \le \alpha_t(\omega) + \frac{\gamma}{2}|z|^2 .$$

3) $Ee^{\gamma\eta + \gamma\int_0^T \alpha_s dC_s} < \infty$ *and* $\eta + \int_0^T f(s, 0)dC_s \ge -D$ *for some* $D \ge 0$.

Then there exists a solution $V = \{V_t\}_{0\le t\le T}$ *of equation* (1), *which is represented in the form:*

$$V_t = \operatorname*{ess\,sup}_{u\in U} E\left[\mathcal{E}_{t,T}\left(\int f'(u)dM\right)\left(\eta + \int_t^T [f(s,u_s) - f'(s,u_s)u_s]dC_s\right)\Big| F_t\right]$$

where U is the class of predictable, bounded R^d*-valued controls:*

$$U = \{u : |u_t| \le C_u \quad \text{for a constant} \quad C_u \ge 0\}.$$

The main idea of the proof is the following: Because f is convex we have the equality:

$$f(t, Z_t) = \operatorname*{ess\,sup}_{u\in U}[f(t,u_t) + f'(t,u_t)(Z_t - u_t)].$$

Let us consider the linear (BSDE):

$$\begin{cases} Y_t = Y_0 - \int_0^t [f(s,u_s) + f'(s,u_s)(Z_s - u_s)]dC_s + \int_0^t Z_s dM_s + L_t \\ Y_T = \eta \end{cases} \tag{2}$$

For any $u \in U$ equation (2) admits the unique solution (Y^u, Z^u, L^u) where the first component Y^u has the form:

$$Y_t^u = E^u\left[\eta + \int_t^T [f(s,u_s) - f'(s,u_s)u_s]dC_s \Big| F_t\right].$$

Here E^u denotes the conditional expectation with respect to measure $dP^u = \mathcal{E}_T\left(\int f'(u)dM\right)dP$. Our task is to prove that the value process $V_t = \operatorname*{ess\,sup}_{u\in U} Y_t^u$ is the solution of equation (1).

We shall show this first in the case when η, C_T and $\int_0^T \alpha_s^2 dC_s$ are bounded random variables and then we complete the proof of the Theorem by passing to the limit.

The Linear Backward Stochastic Differential Equations (LBSDEs) were first considered by M. Bismut [1]. R. Chitashvili derived the stochastic Bellman equation in non-Markov case and solved a non-linear (BSDE) with Bellman generator [2]. E. Pardoux and S. Peng introduced the general BSDE and proved the existence and uniqueness of the solution when the generator is a Lipschitz function [3].

M. Kobylanski [4], J. P. Lepeltier and J. San Martin [5] in Brownian case proved the existence of the solution when the generator f has the quadratic growth with respect to z. These results were generalized by R. Tevzadze [6] and M. A. Morlais [7] for BSDEs driven by continuous martingales with bounded characteristic. P. Briand and Y. Hu [8] generalized Kobylanski's result proving the existence of a solution with unbounded terminal condition in Brownian case. F. Delbaen, Y. Hu and A. Richou [9] considered the BSDEs in Brownian setting with convex generators of quadratic growth. Using the Legandre-Frencher transform of convex functions they showed that any solution of such equation can be expressed as a value function of a certain optimization problem, which proves the uniqueness of the solution.

The main novelty of this article is the proof of the existence of a solution for BSDEs driven by continuous martingales with an unbounded characteristic, where we show that the value function of a certain optimization problem satisfies equation (1). Note that our optimization problem and the class of admissible strategies are different with the problem defined in [9], but the resulting value functions will coincide (if we take the opposite sign in the generator). In contrast to [9] we use a linear envelope of convex functions and the optimal strategy of our problem does not exist in general. This can be viewed as a method of proving that the value process related to a non-Markov optimization problem satisfies a corresponding stochastic Bellman equation, in the case when an optimal control may not exist.

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