საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 5, №3, 2011
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 5, no. 3, 2011

Mathematics

Factorization of Loops in Loop Groups

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(Presented by Academy Member Revaz Gamkrelidze)

ABSTRACT. We deal with loops in the loop group of a compact Lie group. In this context, we obtain generalizations of several results on existence of Birkhoff factorization for matrices with parameters and outline their applications to the Riemann-Hilbert problem in loop spaces discussed in preceding papers of the authors. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *Birkhoff factorization, partial indices, Riemann-Hilbert problem, loop group.*

A centerpiece of the classical factorization theory (in other terminology: vector Riemann-Hilbert problem, see, e.g., the monographs [1, 2]) is the existence of representation

$$A(t) = A_+(t)\Lambda(t)A_-(t), \quad t \in T \tag{1}$$

for any sufficiently smooth $n \times n$ non-singular matrix function A defined on the unit circle $\mathbf{T}$. Here $A_+(t)$ and $A_-(t)$ are square matrix functions, analytic and invertible inside and outside $\mathbf{T}$ and continuous up to $\mathbf{T}$, while Λ is a diagonal matrix with the diagonal entries of the form t^{κ_j}, $\kappa_j \in \mathbb{Z}$, $j = 1,\dots,n$. The integer parameters of factorization are defined by A uniquely, up to their permutation, and are called its *partial indices*.

An algebraic geometrical interpretation of this result is Grothendieck's theorem according to which every n-dimensional holomorphic bundle over the projective line CP^1 (i.e. vector bundle where every fiber is a copy of $\mathbb{C}^n$ and transition mappings are holomorphic) is isomorphic to a direct sum of n holomorphic line bundles (unique up to permutations) [3]. The classical results on existence of factorization were generalized for matrix-functions G in various ways, in particular, the existence of similar factorization was shown for certain classes of matrix functions depending on parameters (see, e.g., [4, 5]). Given such a factorization it is easy to find the index of the corresponding homogeneous Riemann-Hilbert problem and construct a basis for its solution space [1]. Thus the classical matrix factorization theory provides a key tool for solving the vector Riemann-Hilbert problem and its various modifications and generalizations described in [2].

Another important manifestation of the matrix factorization paradigm (in this setting usually called *Birkhoff factorization*) emerges in the framework of loop groups theory and classification of principal G-bundles with G being a compact Lie group [8]. More precisely, it turns out that each sufficiently smooth loop f in a compact Lie group G of rank r admits a factorization similar to (1) which defines an unordered r-tuple of integers called G-*indices* of f. The classical case corresponds to $G = U(n)$, the unitary group. Birkhoff factorization has important applications to the theory of instantons [6] and Riemann-Hilbert problems with coefficients in compact Lie group [7].

The setting and results of [3] suggest that it may be reasonable to consider analogs of the Riemann-Hilbert problem (RHP) and factorization problem (FP) for functions with values in any Lie group having a natural complex structure. With this paradigm in mind, certain analogs of Riemann-Hilbert problem and factorization problem for mappings with values in based loop group BG of a compact Lie group G have been formulated and studied in [7]. Later on, some existence theorems for solutions to the aforementioned RHP with values in BG have been obtained in [8] as corollaries to certain deep results of [6]. However, the results in [6] and [8] were formulated for moduli spaces of solutions with respect to gauge equivalence and did not provide any effective way for solving RHPs with values in BG. In particular, no analogs of Birkhoff factorization have been considered in this context.

The aim of the present note is to describe a natural connection of the factorization problem in BG with the aforementioned results on matrix factorization with parameters and show how one can use this connection to obtain existence results for factorization of BG-valued functions and for solutions of BG-valued RHPs. We now give the exact formulation of the problem and main results.

Let G be a compact Lie group of rank r. Denote by BG the group of based differentiable loops in G endowed with the complex structure J defined by the Hilbert transform as explained in [3]. Let F be a based differentiable loop in BG, i.e. a differentiable mapping from $\mathbf{T}$ to BG. We are interested in finding a representation of the form

$$F(s) = F_+(s)DF_-(s), \quad s \in T \tag{2}$$

where in the left hand side we take the pointwise multiplication of loops, $F_\pm$ are restrictions to $\mathbf{T}$ of certain holomorphic maps of the unit disc and its complement in the Riemann sphere respectively, and D is a homomorphism of the torus $\mathbf{T}^2$ into a certain maximal torus of G. Since each ingredient in the above formula is a loop in BG, we can put $F(s,t) = F(s)(t)$ and rewrite the equation above as

$$F(s,t) = F_+(s,t)diag(t^k)F_-(s,t), \quad s,t \in T \tag{3}$$

It is convenient to distinguish the two copies $\mathbf{T}_s$, $\mathbf{T}_t$ of $\mathbf{T}$, where the index indicates which one of the variables s, t takes values in the corresponding copy.

In this situation the link between the two settings is given by a simple but important observation formulated as a lemma which can be proved in a quite straightforward way.

Lemma 1. *The mappings F_+ and F_- from the equation* (3) *are holomorphic as mappings into the based loop group BG endowed with the complex structure J described above*.

The importance of this lemma lies in the fact that it enables one to construct solutions to a BG-valued RHP defined by the loop F: $\mathbf{T}_s \to BG$ similarly to the classical case. Notice now that, by a way of analogy with the aforementioned Grothendieck theorem, such a loop defines a principal bundle E_F over CP^1 with the structural group BG. For such bundles one can introduce the concept of deformation in the framework of the general theory of principal bundles (see, e.g., [4]) and say that a bundle of the form E_F is stable if each of its small deformations is isomorphic to it as a holomorphic BG-bundle. Since stable BG-bundles naturally arise in certain problems of mathematical physics (see, e.g., [3]) it is natural to look for sufficient conditions of stability of E_F in terms of factorization (3).

It turns out that one can derive many results about BG-valued RHP and BG-bundles using the established connection with the matrix factorization with parameters. Namely, combining our lemma and results of [4, 5] in the way described in [8], we arrive at the an important result on the existence of Birkhoff factorization in BG which, in particular, yields a sufficient condition for stability of E_G.

Theorem 1. *Let G be a compact Lie group and BG its group of differentiable based loops. Then each loop F in the group BG such that, for each $t \in \mathbf{T}$, the G-indices of the loop $F(t) \in LG$ are constant, there exists a representation $F = F_+DF_-$, where D is a homomorphism in a certain maximal torus $\mathbf{T}^r$ of G with the exponents equal to G-indices of F, and F_+, F_- are holomorphic in the interior and exterior of the unit disc respectively.*

Corollary 1. *Let F be a differentiable loop in the loop group LG of G. If all G-indices of F(t) vanish then F admits a canonical factorization $F = F_+DF_-$, where F_+, F_- are holomorphic in the interior and exterior of the unit disc respectively*.

Using the canonical factorization by the usual scheme (see, e.g., [2]) we become able to construct explicit solutions to the corresponding RHP, which, in particular, implies solvability in this case.

Theorem 2. *Let G be a compact Lie group and BG its group of differentiable based loops. If a loop F is such that* $F(t) \in LG$ *has vanishing G-indices then the Riemann-Hilbert problem with values in BG and coefficient F is solvable.*

Taking into account the connection with *BG*-bundles described above and results of [4], one can use the above corollary to obtain also a sufficient condition of stability for E_F.

Theorem 3. *Let F be a differentiable loop in the loop group LG of G such that all G-indices of F(t) vanish for each* $t \in \mathbf{T}$. *Then the principal BG-bundle* E_F *on* CP^1 *with the transition function F is stable.*

In fact one can use the above results to describe the structure of solutions as well. Namely, taking into account the connections described above and applying the arguments used in [8] one can obtain an explication (and another proof) of the main result of [8] (cf. [6]).

Theorem 4. *Let G be a compact Lie group and BG its group of differentiable based loops. If a loop F is such that* $F(t) \in LG$ *has vanishing G-indices then the Riemann-Hilbert problem with values in BG and coefficient F is solvable and the moduli space of its solutions is homeomorphic to the moduli space of G-instantons on* $\mathbb{R}^4$.

It would be interesting to obtain analogs of the above results in the case of non-vanishing indices. It would be also able to consider similar problems in the so-called unparameterized loop space of SU(2) endowed with the so-called Brylinski almost complex structure [8]. The point is that this structure is non-integrable so very few is known about solutions to RHP with values in this space.

As will be shown in our next publications, the results of this note are closely related to recent applications of factorization theory to the theory of integrable systems [6]. They also have further applications to the the Riemann-Hilbert problem in loop spaces considered in [7-9].

Acknowledgement. The research of the first author was supported by the Shota Rustaveli National Science Foundation Grant No. 1-3/85. The second author acknowledges financial support of the International Center of Theoretical Physics (ICTP) during his visits to Trieste in 2011 where some of the above results have been obtained.

მათემატიკა

მარყუჟების ფაქტორიზაცია მარყუჟების ჯგუფში

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(წარმოდგენილია აკადემიის წევრის რ. გამყრელიძის მიერ)

ნაშრომში განხილულია მარყუჟები მნიშვნელობებით მარყუჟების ჯგუფში, როდესაც ჯგუფი კომპაქტურია. მიღებულია რიგი შედეგები ამ კონტექსტში, კერძოდ, ბირკგოფის ფაქტორიზაციის ანალოგი პარამეტრზე დამოკიდებული მატრიცული ფუნქციებისათვის და მითითებულია მათი გამოყენებები განზოგადებული რიმან-ჰილბერტის ამოცანის შესწავლისათვის, რომელიც დაწყებული იყო ჩვენს ადრინდელ ნაშრომებში.

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Received September, 2011