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Physics

Calculation of the Trajectory of a Test Particle in the FRW Spacetime Based on Lyra Geometry for the Perfect Fluid with Massless Scalar Field

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ABSTRACT. The exact solutions of the field equations for the FRW cosmological model are presented when the source of gravitational field is a perfect fluid coupled with a massless scalar field within the framework of Lyra geometry. Then, the trajectory of a test particle in this spacetime by using the Hamilton-Jacobi formalism is calculated. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *FRW spacetime, Lyra geometry, perfect fluid, massless scalar field.*

1. Introduction

Einstein provided a general theory of gravitation by geometry. This theory has been very successful in describing the gravitational phenomena. Einstein field equations without the cosmological constant admitted only nonstatic solutions and he introduced the cosmological constant in order to obtain the static models. The properties of the spacetime require the Riemannian geometry for their description. Several modifications of Riemannian geometry have been suggested to unify gravitation, electromagnetism and other effects in universe. One of the modified theories has been introduced by Lyra [1]. He introduced an additional gauge function into the structureless manifold as a result of which a displacement vector field arises naturally from geometry. The Einstein field equations in normal gauge based on Lyra geometry defined by Sen [2] and Sen and Dunn [3] as

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R+\frac{3}{2}\left(\xi_{\alpha}\xi_{\beta}-\frac{1}{2}g_{\alpha\beta}\xi_{\mu}\xi^{\mu}\right)=T_{\alpha\beta}\,, \tag{1}$$

where ξ_{α} is the Lyra displacement vector field, other symbols have their usual meaning as in Riemannian geometry and gravitational units with $8\pi G=c=1$ are used. In Lyra formalism, the constant displacement field plays the same role as the cosmological constant in the standard general relativity [4]. In this paper, we will determine the trajectory of a moving test particle in the FRW spacetime in this model.

2. The metric and field equations

We assume that the metric of the spacetime, in which the cosmic fluid resides, is the spatially flat FRW metric, which is favored by present observations [5], with the following line element

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3}\left(dx^i\right)^2 , \tag{2}$$

where $a(t)$ is the scale factor of universe. In this analysis, we assume that the matter content is a perfect fluid endowed with a massless scalar field, i.e. the energy-momentum tensor is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} + \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha} , \tag{3}$$

where ρ, p and u_μ are respectively the matter density, isotropic pressure and four velocity vector of the matter distribution with co-moving coordinates as $u_\alpha = (1, 0, 0, 0)$ and ϕ is the massless scalar field such that satisfies the Klein-Gordon equation as

$$\Box\phi = 0 , \tag{4}$$

where $\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right]$ while $g = \det(g_{\mu\nu})$. By considering $\phi = \phi(t)$, the equation (4) changed to the following relation

$$\ddot{\phi} + 3H\dot{\phi} = 0 , \tag{5}$$

where H is the Hubble parameter and overdot means differentiation with respect to the time. Next, the solution of this equation is

$$\dot{\phi} = e^{-3\int H dt} . \tag{6}$$

To continue our analysis, we consider the Lyra displacement vector to be time-like vector as

$$\xi_\mu = (\lambda, 0, 0, 0) , \tag{7}$$

where λ is a function of time. In the next step, the field equations (1) together with (6) for the FRW metric lead to the following set of equations

$$6H^2 - 2\rho - e^{-6\int H dt} + \frac{3}{2}\lambda^2 = 0 , \tag{8}$$

$$6H^2 + 4\dot{H} + 2p + e^{-6\int H dt} - \frac{3}{2}\lambda^2 = 0 , \tag{9}$$

where the quantities ρ and p depend on time only. On the other hand, the energy momentum tensor satisfies the energy conservation law as

$$\mathrm{T}^{\mu\nu}_{;\nu} = 0 , \tag{10}$$

here semicolon denotes the covariant differentiation. After some work, this equation becomes

$$\dot{\rho} + 3H(\rho + p) = 0 . \tag{11}$$

Therefore, there are only three field equations containing four unknowns and so we need to more relation. Hence, let us now assume that ρ and p satisfy the following simple EoS:

$$p = \omega\rho, \tag{12}$$

and if ω be a constant, the equation (11) is changed to

$$\rho^{\frac{1}{1+\omega}} = e^{-3\int H dt} . \tag{13}$$

Now we are going to study the power law solution as

$$a = a_0 t^n, \quad H = \frac{n}{t}, \tag{14}$$

where n and a_0 are constants. Therefore, form equation (8) and (9) we can conclude

$$\rho = \frac{3n^2}{t^2} - \frac{1}{2t^{6n}} + \frac{3}{4}\lambda^2, \tag{15}$$

$$p = \frac{n(2-3n)}{t^2} - \frac{1}{2t^{6n}} + \frac{3}{4}\lambda^2. \tag{16}$$

Also, by substituting of equation (14) into equation (13), we obtain

$$\rho = \rho_0 t^{-3n(1+\omega)}, \tag{17}$$

here ρ_0 is a constant and without loss of generality we consider $\rho_0 = 1$. In the next step, by comparing equations (15), (16) and (17), we deduce

$$\omega = \frac{2-3n}{3n}, \tag{18}$$

$$3n^3 - n^2 - n + \frac{1}{3} = 0. \tag{19}$$

The last equation has three real roots given by $n = \frac{1}{3}$ and $\pm\frac{1}{\sqrt{3}}$. Here we discuss these two cases as follows:

Case I. $n = \frac{1}{3}$.

In this case we have $\omega = 1$ (i.e. Zeldovich fluid), $\rho = \frac{1}{t^2}$, $\phi = \ln t$ and $\lambda = \pm\frac{\sqrt{14}}{3t}$.

Case II. $n = \pm\frac{1}{\sqrt{3}}$.

In this case we have $\omega = 2n - 1$, $\rho = \frac{1}{t^2}$, $\phi = \frac{n}{n-1} t^{\frac{n-1}{n}}$ and $\lambda = \pm\sqrt{\frac{2}{3}} t^{-\frac{1}{n}}$.

3. Calculation of the trajectory of a test particle in the FRW metric

Below we will determine the trajectory of a test particle with mass m that moving in the flat FRW spacetime by using the Hamilton-Jacobi equation [6-8]. Thus, we have

$$a_0^{-2} t^{-2n} \left[\left(S_{,x}\right)^2 + \left(S_{,y}\right)^2 + \left(S_{,z}\right)^2 \right] - \left(\dot{S}\right)^2 + m^2 = 0. \tag{20}$$

For solving this partial differential equation, we use the method of separation of variables for the Hamilton-Jacobi function as follows

$$S(x,y,z,t) = \ell_x x + \ell_y y + \ell_z z + f(t), \tag{21}$$

where ℓ_x, ℓ_y and ℓ_z are arbitrary constants and can be identified as the angular momentum of test particle along x, y and z-directions. With substituting the ansatz (21) in Hamilton-Jacobi equation, the unknown function $f(t)$ is given by

$$f(t) = mt\begin{cases} \left(1+bt^{-\frac{2}{3}}\right)^{\frac{3}{2}} & n=\frac{1}{3}, \\ \mathrm{F}\left(-\frac{1}{2},-\frac{1}{2n},\frac{2n-1}{2n};-bt^{-2n}\right) & n=\pm\frac{1}{\sqrt{3}}, \end{cases} \tag{22}$$

where $b=\dfrac{\ell_x^2+\ell_y^2+\ell_z^2}{m^2 a_0^2}$ and F is the generalized hypergeometric function. For more details about generalized hypergeometric functions, see references [9-11]. Let us now obtain the trajectory of test particle by considering the following relations [6-8]:

$$\frac{\partial S}{\partial \ell_x} = \text{constant}, \quad \frac{\partial S}{\partial \ell_y} = \text{constant}, \quad \frac{\partial S}{\partial \ell_z} = \text{constant}, \tag{23}$$

without loss of generality one can consider the above constants to be zero. Consequently, after calculating and simplifying, the set of equations (23) change to the following relation

$$\frac{x}{\ell_x}=\frac{y}{\ell_y}=\frac{z}{\ell_z}=\frac{1}{ma_0^2}\begin{cases} 3t^{\frac{1}{3}}\left(1+bt^{-\frac{2}{3}}\right)^{\frac{1}{2}} & n=\frac{1}{3}, \\ \frac{1}{1-2n}t^{1-2n}\mathrm{F}\left(\frac{1}{2},\frac{2n-1}{2n},\frac{4n-1}{2n};-bt^{-2n}\right) & n=\pm\frac{1}{\sqrt{3}}. \end{cases} \tag{24}$$

Therefore, the trajectory of test particle is calculated.

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უწონადი სკალარული ველის მქონე ტესტური ნაწილაკის ტრაექტორიის გამოთვლა იდეალური სითხეებისათვის ლირას გეომეტრიის FRW-დრო-სივრცეში

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მოცემულია ველის განტოლებების ზუსტი ამონახსნები FRW-კოსმოლოგიური მოდელისათვის ლირას გეომეტრიის ბაზაზე, როდესაც გრავიტაციული ველი წარმოადგენს იდეალურ სითხეს უწონადი სკალარული ველით. გამოთვლილია საცდელი ნაწილაკის ტრაექტორია ჰამილტონ-იაკობის მოდელის გამოყენებით.

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