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*Mathematics*

# On the $\mathcal{D}$-equivalence Class of a Graph

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**ABSTRACT. Let $G$ be a simple graph of order $n$. The domination polynomial of $G$ is the polynomial $D(G,x)=\sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G, i)$ is the number of dominating sets of $G$ of size $i$, and $\gamma(G)$ is the domination number of $G$. A dominating set with cardinality $\gamma(G)$ is called a $\gamma$-set. Two graphs $G$ and $H$ are said to be $\mathcal{D}$-equivalent, written $G \sim H$, if $D(G, x) = D(H, x)$. The $\mathcal{D}$-equivalence class of $G$ is $[G]=\{H : H \sim G\}$. A graph $G$ is said to be $\mathcal{D}$-unique, if $[G] = \{G\}$. In this paper we study the $\mathcal{D}$-equivalence of some graphs. Also, we obtain some properties of graphs with unique $\gamma$-set.** *© 2012 Bull. Georg. Natl. Acad. Sci.*

***Key words:*** *domination polynomial, equivalence.*

## 1. Introduction

Let $G=(V,E)$ be a simple graph. The order of $G$ is the number of vertices of $G$. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v)=\{u \in V \mid uv \in E\}$ and the closed neighborhood is the set $N[v]=N(v)\cup\{v\}$. For a set $S \subseteq V$, the open neighborhood is $N(S)=\bigcup_{v\in S} N(v)$ and the closed neighborhood is $N[S]=N(S)\cup S$. A set $S \subseteq V$ is a dominating set if $N[S]=V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. A dominating set with cardinality $\gamma(G)$ is called a $\gamma$-set. For a detailed treatment of this parameter, the reader is referred to [2]. Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph $G$ with cardinality $i$ and let $d(G, i)=|\mathcal{D}(G, i)|$. The domination polynomial $D(G, x)$ of $G$ is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, [1].

The following theorem follows from the definitions of isomorphic graphs and domination polynomial.

**Theorem 1.** *If $G$ and $H$ are isomorphic, then $D(G, x) = D(H, x)$.*

The converse of the above theorem is not true. There are numerous graphs with the same domination polynomials. Two graphs $G$ and $H$ are said to be *dominating equivalence*, or simply $\mathcal{D}$-equivalent, written $G$

~ $H$, if $D(G,x)=D(H,x)$. It is evident that the relation ~ of being $\mathcal{D}$-equivalence is an equivalence relation on the family $\mathcal{D}$ of graphs, and thus $G$ is partitioned into equivalence classes, called the *$\mathcal{D}$-equivalence classes*. Given $G \in \mathcal{G}$, let $[G]=\{H \in \mathcal{G}: H \sim G\}$. We call $[G]$ the equivalence class determined by $G$. A graph $G$ is said to be *dominating unique*, or simply $\mathcal{D}$-unique, if $[G]=\{G\}$. Determining $\mathcal{D}$-equivalence class of graphs is one of the interesting problems on equivalence classes. Fig. 1 shows all connected graphs of order 5 with the same domination polynomials. Note that for disconnected graphs, we can use the following theorem:

**Theorem 2.** [1] *If a graph G has m components* $G_1,\ldots,G_m$, *then* $D(G,x)=D(G_1,x)\ldots D(G_m,x)$.

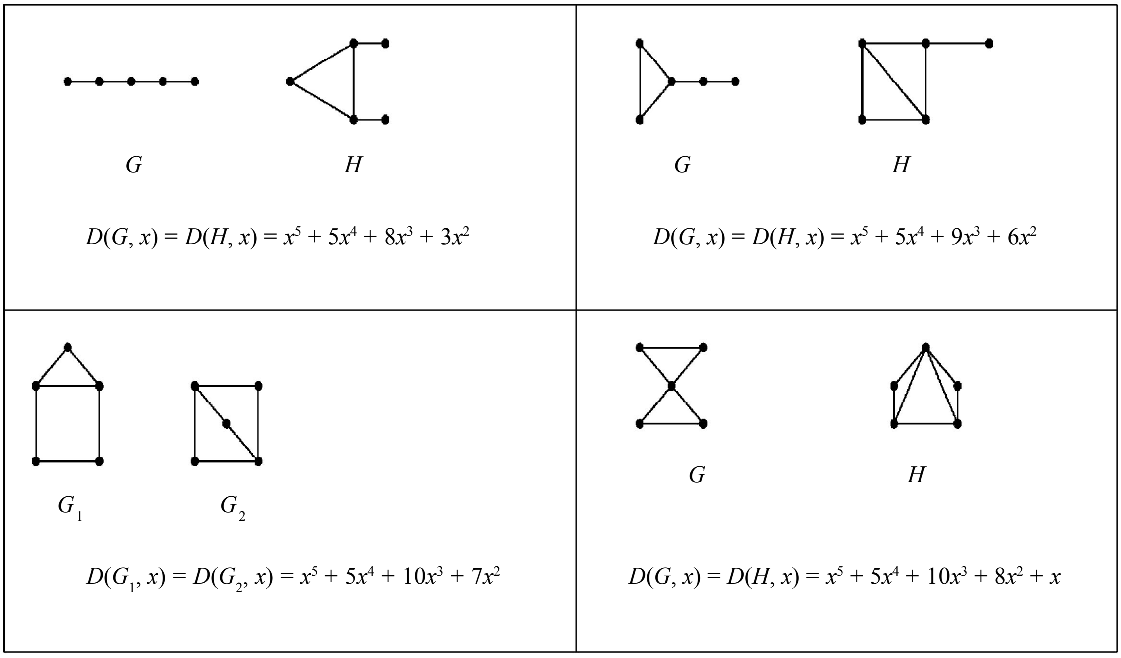


**Fig. 1.** Graphs of order 5 with identical domination polynomials.

The *join* of two graphs $G_1$ and $G_2$, denoted by $G_1+G_2$ is a graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)\cup\{uv \mid u\in V(G_1) \text{ and } V(G_2)\}$. As usual we denote the complete bipartite graph by $K_{m,n}$, the complete graph, path and cycle of order $n$ by $K_n$, $P_n$ and $C_n$, respectively. Also $K_{1,n}$ is the star graph with $n+1$ vertices.

In Section 2 we study the $\mathcal{D}$-equivalence classes of some specific graphs. In Section 3 we study some properties of graphs with unique $\gamma$-set.

## 2. $\mathcal{D}$-Equivalence classes of some graphs

In this section, we study the $\mathcal{D}$-equivalence class of $K_n$ and $K_{1,n}$. First, we recall the following theorem which gives a formula for the computation of the domination polynomial of join of two graphs.

**Theorem 3.** [1] *Let* $G_1$ *and* $G_2$ *be graphs of orders* $n_1$ *and* $n_2$, *respectively. Then*

$$D(G_1+G_2,x)=\left((1+x)^{n_1}-1\right)\left((1+x)^{n_2}-1\right)+D(G_1,x)+D(G_2,x).$$

**Theorem 4.** *Assume that G is a graph of order n and* $v\in V(G)$. *If* $deg(v)=n-1$, *then G is* $\mathcal{D}$-*unique, if and only if* $G\backslash\{v\}$ *is* $\mathcal{D}$-*unique. Hence* $K_n$ *and* $K_{1,n}$ *are* $\mathcal{D}$-*unique for every natural number n.*

**Proof.** By Theorem 3, $D(G,x) = x\left((1+x)^{n-1}-1\right)+x+D(G\setminus\{v\},x)$. Thus $G$ is $\mathcal{D}$-unique if and only if $G\setminus\{v\}$ is $\mathcal{D}$-unique.

**Remark.** The $\mathcal{D}$-equivalence class of $K_{n,n}$ is not unique. We can see this as follows. Consider two disjoint copies of complete graph $K_n$ with vertex sets $\{v_1,\dots,v_n\}$ and $\{v_1',\dots,v_n'\}$, respectively. Join $v_i$ to $v_i'$ for every $i$, $1 \le i \le n$. It can be easily seen that the domination polynomial of this graph is the same as the domination polynomial of $K_{n,n}$.

## 3. Some graphs with unique $\gamma$-set

**Lemma 1.** *Let G be a graph and u be a pendant vertex of G. Suppose that* $uv \in E(G)$, *deg*$(v) = 2$ and $N(v) = \{u,w\}$. *If G has a unique* $\gamma$*-set, then* $G\setminus\{u,v,w\}$ *has a unique* $\gamma$*-set.*

**Proof.** Assume that $\gamma(G) = t$, and $D \subseteq V(G)$ is a unique $\gamma$-set of $G$ with size $t$. Since $u$ is a pendant vertex, either one of the vertices $u$ or $v$ should be contained in $D$. Obviously, $v \in D$, for otherwise $(D\setminus\{u\})\cup\{v\}$ is another $\gamma$-set for $G$, a contradiction. Clearly, $w \notin D$, for otherwise $(D\setminus\{v\})\cup\{u\}$ is another $\gamma$-set for $G$, a contradiction. Hence $|D\cap(G\setminus\{u,v,w\})| = t-1$. We claim that $\gamma(G\setminus\{u,v,w\}) = t-1$. If $\gamma(G\setminus\{u,v,w\}) < t-1$, then by adding $v$ to a $\gamma$-set for $G\setminus\{u,v,w\}$ we obtain a $\gamma$-set for $G$ with size at most $t-1$, a contradiction. Thus $\gamma(G\setminus\{u,v,w\}) = t-1$. For every $\gamma$-set $S$ of $G\setminus\{u,v,w\}$, $S\cup\{v\}$ is a $\gamma$-set of $G$. This implies that $G\setminus\{u,v,w\}$ has a unique $\gamma$-set and the proof is complete.

We have the following corollary for path $P_n$ by Lemma 1:

**Corollary 1.** *If* $n \equiv 0 \pmod 3$, *then* $P_n$ *has a unique* $\gamma$*-set.*

By Lemma 1, we have the following corollaries for the graphs $C_n(m)$ and $T_{n_1,n_2,n_3}$ (see Fig. 2):

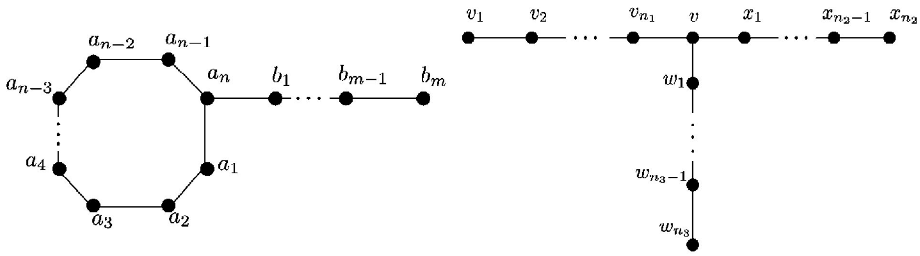


**Fig. 2.** The graphs $C_n(m)$ and $T_{n_1,n_2,n_3}$, respectively.

**Corollary 2.** *Suppose that m and n are natural numbers. The graph* $C_n(m)$ *has a unique* $\gamma$*-set, if and only if one of the following holds*:

(*i*) $n \equiv 0 \pmod 3$ *and* $n \equiv 1 \pmod 3$,

(*ii*) $n \equiv 1 \pmod 3$ *and* $n \equiv 2 \pmod 3$.

**Corollary 3.** *Suppose that* $n_1$; $n_2$ *and* $n_3$ *are natural numbers. If* $n_1 \equiv n_2 \equiv n_3 \equiv 1 (mod\ 3)$, *then the tree* $T_{n_1,n_2,n_3}$ *has a unique* $\gamma$*-set.*

**Lemma 2.** *Let G be a graph,* $u,v \in V(G)$ *and deg*$(u)$ = *deg*$(v) = 1$. *If* $uw, vw' \in E(G)$ *and* $ww' \notin E(G)$, *then* $D(G,x) = D(G+ww',x)$, *and hence G is not* $\mathcal{D}$*-unique.*

**Proof.** Clearly, every dominating set for G is a dominating set for $G+ww'$. Now, let $S \subseteq V(G)$ be a dominating set for $G + ww'$. If both $w, w' \in S$ or both $w, w' \notin S$, then obviously $S$ is also a dominating set for $G$. So suppose that $w \in S$ and $w' \notin S$ (or $w \notin S$ and $w' \in S$). Since $S$ is a dominating set for $G + ww'$, we have v 2 S. This implies that $S$ is a dominating set for $G$. Therefore we conclude that $D(G, x) = D(G + ww', x)$ and the proof is complete.

The following theorem is an immediate conclusion of Lemma 2:

**Theorem 5.** *For every $n \geq 5$, $P_n$ is not $\mathcal{D}$-unique, and $[P_n]$ contains at least two following graphs:*


**Fig. 3.** Two graphs of $[P_n]$.

*მათემატიკა*

# გრაფის $\mathcal{D}$-ეკვივალენტურობის კლასების შესახებ

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**ვთქვათ $G$ $n$-რიგის მარტივი გრაფია. $G$-ს დომინატური მრავალწევრი ეწოდება $D(G, x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i$ გამოსახულებას, სადაც $d(G,i)$ არის $G$-ს $i$ სიგრძის დომინატური სიმრავლეების რიცხვი, ხოლო $\gamma(G)$ კი $G$-ს დომინირების რიცხვი. $\gamma(G)$-ს ტოლი დომინირების რიცხვის მქონე დომინატურ სიმრავლეს ეწოდება $\gamma$-სიმრავლე. ორ $G$ და $H$ გრაფს ეწოდება $\mathcal{D}$-ეკვივალენტური და იწერება $G \sim H$, თუ $D(G,x)=D(H,x)$. $G$-ს $\mathcal{D}$-ეკვივალენტურობის კლასია $[G]=\{H : H \sim G\}$. $G$ გრაფს ეწოდება $\mathcal{D}$-ერთადერთი, თუ $[G]=\{G\}$. ნაშრომში შესწავლილია გარკვეული ტიპის გრაფების $\mathcal{D}$-ეკვივალენტურობა. აგრეთვე, მიღებულია ერთადერთი $\gamma$-სიმრავლის მქონე გრაფების ზოგიერთი თვისება.**

## REFERENCES

1. *S. Akbari, S. Alikhani and Y.H. Peng* (2010), Euro.J.Combin., **31**: 1714-1724.
2. *T. W. Haynes, S. T. Hedetniemi, P. J. Slater* (1998), Fundamentals of Domination in Graphs. Marcel Dekker, NewYork.

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