საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 6, №1, 2012
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 6, no. 1, 2012

Physics

An Investigation of Bound *qqq*-Systems on the Basis of Salpeter Equation in the Framework of Simple Approach with Use of Expansion in Terms of Hyperspherical Harmonics

Tamar Babutsidze*, Teimuraz Kopaleishvili, Vazha Skhirtladze***

* *Institute of High Energy Physics, I. Javakhishvili Tbilisi State University*
***Academy Member, Institute of High Energy Physics, I. Javakhishvili Tbilisi State University*

ABSTRACT. The approach is developed to the solution of a problem of three bound constituent quarks (baryon) on the basis of Salpeter equation with two required 8-component spinors, having clear physical sense in the meaning of a particle-antiparticle (baryon-antibaryon), without so-called relativization of full wave function. The doubtful character of consideration of two-particle interaction under quark confinement conditions is stressed. It is proposed to use expansion in terms of hyperspherical harmonics for calculations of compact bound systems with three-particle interactions. Two elementary types of the central three-particle interaction - linear and oscillatory potentials - are considered. The approach proposed in this paper will be applied numerically to light baryon calculations. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *bound three-quarks systems, baryons, Salpeter equation, 8-component spinors, three-particle linear and oscillatory potentials, hyperspherical harmonics.*

Advances in nonrelativistic quark models for description of the bound states of *qqq*-systems are well-known [1]. However relatively large binding energies of light baryons (N, Σ, Λ, Ξ, Δ, Ω), consisting of constituent quarks (u, d, s), lead to the necessity of accounting for relativism. The relativistic-covariant approach is realised within the framework of Bethe-Salpeter equation [2] generalized to three particles [3]. The homogeneous integral B-S equation for the bound states of quarks is derived from the first principles – the 4-dimensional six-point Green's function has a pole at energy equal to bound state (baryon) mass [4-6]. Therefore quark confinement (lack of the free spectator quark, or lack of asymptotic states of the inhomogeneous B-S equation) does not interfere with the statement of the equation for three bound quarks, rather it specifies the basic role of three-particle interactions in baryons. However, problems related to probabilistic interpretation and normalization of a wave function arise. The absurdity of wave function dependence on relative time of particles is precisely enough and wittily described by the following expression: "Electron today and proton tomorrow do not form the bound state – a hydrogen atom" (see e.g. [7]). As with two particles [6], these problems are overcome at the instantaneous approximation excluding the B-S equation kernel dependence on relative energy variables in momentum space. It is very important that the obtained 3-dimensional Salpeter equation remains relativistic-invariant [8] and the wave function gains usual probabilistic sense. At the same time, in a coordinate space the instantaneous approximation arranges all three quarks on a spacelike hypersurface, interquark interaction takes a potential character, i.e. interaction propagates with infinite velocity and effects of retardation are formally missing. The conventional two-particle forces (considered for either reasons in the kernel of 4-dimensional B-S equation [8]) in Salpeter equation have the formal character, because of quark confinement they really are three-particle. At first it is necessary to make the instantaneous approximation in the kernel of 4-dimensional B-S equation, and then to talk (or not to talk!) about multiparticleness of interquark interactions, rather than the reverse. The principal virtues of Salpeter equation – relativistic invariance and simultaneous affinity of potential reviewing to a nonrelativistic picture, make this equation especially attractive for the description of baryons. At the same time the present state of QCD does not give the possibility of constructing the kernel of B-S equation and consequently we are forced to choose it phenomenologically.

The basic role of three-particle forces in calculations of baryons together with phenomenologycal choice of Salpeter equation's kernel pushes us to use expansions (for required wave functions) in terms of hyperspherical harmonics (HH) most natural for this case [9-12]. In particular, using solutions of a nonrelativistic Schrödinger equations with oscillatory three-particle potential along with simplification of analytical calculations we hope to achieve fast convergence in specific numerical calculations. In a sense it is possible to consider this paper as prolongation of early examinations for $q\bar{q}$ systems [13-18].

Proceeding from B-S equation for three bound quarks [2, 3] and using an average procedure over energy variables [19], at first we shall come to quasipotential equations and then in instantaneous approximation (in center-of-mass system: $\vec{K}=\vec{k}_1+\vec{k}_2+\vec{k}_3=0$; $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ are quark 3-momenta) we shall obtain Salpeter equations:

$$\left[M+i\varepsilon-h_1(\vec{k}_1)-h_2(\vec{k}_2)-h_3(\vec{k}_3)\right]\Phi_M(\vec{\kappa})=I^{9/4}\hat{\Pi}(\vec{\kappa})\gamma_0^{(1)}\gamma_0^{(2)}\gamma_0^{(3)}\int V(M;\vec{\kappa},\vec{\kappa}')\left[d\vec{\kappa}'/(2\pi)^6\right]\Phi_M(\vec{\kappa}'). \quad (1)$$

In this equation the 6-dimensional vector $\vec{\kappa}=(\vec{q},\vec{p})$ with length $\kappa=\sqrt{q^2+p^2}$ is made up of Jacobi coordinates (in $\vec{\kappa}$ -representation):

$$\left.\begin{aligned}\vec{q}&=\sqrt{(\mu_1+\mu_2)/(\mu_1\mu_2)}(\mu_2\vec{k}_1-\mu_1\vec{k}_2)/(\mu_1+\mu_2),\\ \vec{p}&=[\mu_3(\vec{k}_1+\vec{k}_2)-(\mu_1+\mu_2)\vec{k}_3]/\sqrt{\mu_3(\mu_1+\mu_2)},\end{aligned}\right\} \quad (2)$$

where $\mu_i=m_i/M_0$ ($M_0=m_1+m_2+m_3$) is relative mass of constituent quark. The 6-volume element of momentum space is equal to

$$\left.\begin{aligned}&d\vec{\kappa}=d\vec{q}d\vec{p}=\kappa^5 d\kappa d\Omega_{\vec{\kappa}}\ ,\\ &d\Omega_{\vec{\kappa}}=\cos^2\alpha\sin^2\alpha d\alpha d\Omega_{\vec{q}}d\Omega_{\vec{p}}\ ,\\ &\int d\Omega_{\vec{\kappa}}=\pi^3,\ 0\le\alpha\le\pi/2.\end{aligned}\right\} \quad (3)$$

The projection operator

$$\hat{\Pi}(\vec{\kappa})=\Lambda_1^+(\vec{k}_1)\Lambda_2^+(\vec{k}_2)\Lambda_3^+(\vec{k}_3)+\Lambda_1^-(\vec{k}_1)\Lambda_2^-(\vec{k}_2)\Lambda_3^-(\vec{k}_3) \quad (4)$$

is expressed through usual one-particle projection operators:

$$\Lambda^{\pm}(\vec{k})=[\omega(\vec{k})\pm h(\vec{k})]/[2\omega(\vec{k})]\ , \quad (5)$$

where $h(\vec{k})=\gamma_0\vec{\gamma}\cdot\vec{k}+\gamma_0 m$ is Dirac Hamiltonian of the free quark (γ_0, $\vec{\gamma}$ are Dirac matrices), $\omega(\vec{k})=\sqrt{m^2+\vec{k}^2}$. In the equation (1) $I=1/(\mu_1\mu_2\mu_3)^2$ represents the Jacobian of transformation from Cartesian 4-momenta k_1, k_2 and k_3 to Jacobi 4-momenta K, q and p. The relation between these 4-momenta is similar to 3-dimensional relations (2). Usually the kernel of interaction $V(M;\vec{\kappa},\vec{\kappa}')$ is chosen phenomenologically, mass-independent.

Acting on the equation (1) with operator $\left[M+i\varepsilon-h_1(\vec{k}_1)-h_2(\vec{k}_2)-h_3(\vec{k}_3)\right]^{-1}$ at the left, and using the relations

$$\left.\begin{aligned}&\gamma_0 h(\vec{k})=h(-\vec{k})\gamma_0, \qquad \gamma_0\Lambda^{\pm}(-\vec{k})=\Lambda^{\pm}(\vec{k})\gamma_0,\\ &h(\vec{k})\Lambda^{\pm}(\vec{k})=\pm\omega\Lambda^{\pm}(\vec{k}), \qquad h^2(\vec{k})=\omega^2,\end{aligned}\right\} \quad (6)$$

we shall obtain another form of Salpeter equation:

$$\Phi_M(\vec{\kappa}) = I^{9/4}\{\Lambda_1^+(k_1)\Lambda_2^+(k_2)\Lambda_3^+(k_3)/(M+i\varepsilon-\omega_1-\omega_2-\omega_3)+$$
$$+\Lambda_1^-(k_1)\Lambda_2^-(k_2)\Lambda_3^-(k_3)/(M+\omega_1+\omega_2+\omega_3)\}\times\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)}\int V(M;\vec{\kappa},\vec{\kappa}')[d\vec{\kappa}'/(2\pi)^6]\Phi_M(\vec{\kappa}'). \quad (7)$$

From here, taking into account properties of projection operators $\Lambda^{\pm}(\vec{k})$, it is easy to obtain very useful additional equations:

$$\Phi_M^{++-}(\vec{\kappa}) = \Phi_M^{+-+}(\vec{\kappa}) = \Phi_M^{-++}(\vec{\kappa}) = \Phi_M^{--+}(\vec{\kappa}) = \Phi_M^{-+-}(\vec{\kappa}) = \Phi_M^{--+}(\vec{\kappa}) = 0\,, \quad (8)$$

$$\Phi_M^{\pm\pm\pm}(\vec{\kappa}) = \Lambda_1^{\pm}(\vec{k}_1)\Lambda_2^{\pm}(\vec{k}_2)\Lambda_3^{\pm}(\vec{k}_3)\Phi_M(\vec{\kappa})\,. \quad (9)$$

Further, representing 64-component spinor $\Phi_M(\vec{\kappa})$ in the form of the block column involving eight 8-component spinors Φ_1, Φ_2, ..., Φ_8 and solving six (block) equations (8), it is possible to express the spinors Φ_2, Φ_3, Φ_4, Φ_5, Φ_6 and Φ_7 through two spinors Φ_1 and Φ_8:

$$\left.\begin{aligned}
\Phi_2 &= \{(1-\varepsilon_1\varepsilon_2)\lambda_3\Phi_1+(1+\varepsilon_3)\lambda_1\lambda_2\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),\\
\Phi_3 &= \{(1-\varepsilon_1\varepsilon_3)\lambda_2\Phi_1+(1+\varepsilon_2)\lambda_1\lambda_3\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),\\
\Phi_4 &= \{(1+\varepsilon_1)\lambda_2\lambda_3\Phi_1-(1-\varepsilon_2\varepsilon_3)\lambda_1\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),\\
\Phi_5 &= \{(1-\varepsilon_2\varepsilon_3)\lambda_1\Phi_1+(1+\varepsilon_1)\lambda_2\lambda_3\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),\\
\Phi_6 &= \{(1+\varepsilon_2)\lambda_1\lambda_3\Phi_1-(1-\varepsilon_1\varepsilon_3)\lambda_2\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),\\
\Phi_7 &= \{(1+\varepsilon_3)\lambda_1\lambda_2\Phi_1-(1-\varepsilon_1\varepsilon_2)\lambda_3\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),
\end{aligned}\right\} \quad (10)$$

$$\lambda_i(\vec{k}_i) \equiv \vec{\sigma}_i\cdot\vec{k}_i/[\omega_i(\vec{k}_i)+m_i]\,, \quad \lambda_i\lambda_i = \varepsilon_i\,, \quad \varepsilon_i = (\omega_i-m_i)/(\omega_i+m_i)\,. \quad (11)$$

Thus, determination of 64 components of spinor F_M is reduced to determination of 16 components of two 8-component spinors F_1 and F_8. This circumstance favourably distinguishes Salpeter equation from all other 3-dimensional relativistic equations. The normalizing condition of Salpeter amplitude F_M for the kernel $V(\vec{\kappa},\vec{\kappa}')$ (mass-independent) taking into account relations (10) has the following form:

$$\int[d\vec{\kappa}/(2\pi)^6]\mathscr{N}(\vec{\kappa})\left\{\left|\Phi_1(\vec{\kappa})\right|^2+\left|\Phi_8(\vec{\kappa})\right|^2\right\} = 2M_B, \quad (12)$$

$$\mathscr{N}(\vec{\kappa}) = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)/[I^{1/4}(1+\varepsilon_1\varepsilon_2\varepsilon_3)]. \quad (13)$$

Let us pay attention to one interesting circumstance of sufficiently deep physical sense. As is known [20], the Feynman rule of round of poles $m\to m-i\varepsilon$ ($\varepsilon>0$, $\varepsilon\to 0$)in the full propagator of the free quark leads to a possibility of a motion of particles in time both forward and back. The particle moving back in time is equivalent to an antiparticle moving forward. By the way, this rule maintains the covariance of theory as infinitesimal imaginary addition $-i\varepsilon$ is introduced into invariant mass m. In turn B-S amplitude dependence on relative time is caused by the existence of antiparticles. Motion forward-back in time makes essential configurations for which individual routes in time are different for the bound particles, and so the relative time is large [21, 8]. Projection operator $\hat{\Pi}(\vec{\kappa})$ in Salpeter equation (1) is that "relict" of an average of B-S equation, which corresponds to the contribution of these configurations to bound state amplitude. On the other hand, thanks to this operator in Salpeter equation it is possible to reduce the problem to determination only of two 8-component spinors. Below we shall make clear the physical sense of these spinors.

Charge conjugation $\mathscr{C}$, space parity $\mathscr{P}$ and time-reversal $\mathscr{T}$ operators for baryons are obtained by direct multiplication of corresponding one-particle quark operators and in $\vec{\kappa}$-representation operate as follows:

$$\mathscr{C}\,\Phi_M(\vec{\kappa}) = \gamma_0\gamma_2\cdot\gamma_0\gamma_2\cdot\gamma_0\gamma_2\Phi^*_M(\vec{\kappa})\,, \tag{14}$$

$$\mathscr{P}\,\Phi_M(\vec{\kappa}) = \gamma_0\cdot\gamma_0\cdot\gamma_0\Phi_M(-\vec{\kappa})\,, \tag{15}$$

$$\mathscr{T}\,\Phi_M(\vec{\kappa}) = -i\gamma_1\gamma_3\cdot\gamma_1\gamma_3\cdot\gamma_1\gamma_3\Phi^*_M(-\vec{\kappa})\,. \tag{16}$$

From here we obtain an action $\mathscr{C}\mathscr{P}\mathscr{T}$ -operator on $\Phi_M(\vec{\kappa})$:

$$\mathscr{C}\mathscr{P}\mathscr{T}\,\Phi_M(\vec{\kappa}) = -\gamma_0\gamma_5\cdot\gamma_0\gamma_5\cdot\gamma_0\gamma_5\Phi_M(\vec{\kappa})\,, \tag{17}$$

where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Therefore $\mathscr{C}\mathscr{P}\mathscr{T}$ -symmetry of a strong interaction will be expressed by the following commutation relation (see eq. (1)):

$$\left[\gamma_0\gamma_5\cdot\gamma_0\gamma_5\cdot\gamma_0\gamma_5,\, V(M;\vec{\kappa},\vec{\kappa}')\right] = 0\,. \tag{18}$$

Besides, $\Phi_M(\vec{\kappa})$ solutions of Salpeter equation (1) have also a certain value of parity π since parity operator $\mathscr{P}$ commutes with interaction *V*:

$$\mathscr{P}\,\Phi_{M,\pi}(\vec{\kappa}) = \pi\,\Phi_{M,\pi}(\vec{\kappa})\,. \tag{19}$$

Further in Salpeter equation (1) with negative value of mass $-M < 0$ we shall insert expressions $1 = (\gamma_0\gamma_5\cdot\gamma_0\gamma_5\cdot\gamma_0\gamma_5)(\gamma_0\gamma_5\cdot\gamma_0\gamma_5\cdot\gamma_0\gamma_5)$ before amplitudes $\Phi_{-M,\pi}(\vec{\kappa})$ and $\Phi_{-M,\pi}(\vec{\kappa}')$. Then we shall carry remaining expressions $\gamma_0\gamma_5\cdot\gamma_0\gamma_5\cdot\gamma_0\gamma_5$ up to the end to the left, using a commutator (18) and an anticommutator

$$\{\gamma_0\gamma_5, h(\vec{k})\} = 0\,. \tag{20}$$

The "new" Salpeter equation (obtained as a result of these manipulations) already describes a baryon with the positive mass $M > 0$ and opposite parity $-\pi$:

$$\Phi'_{M,-\pi}(\vec{\kappa}) = \gamma_0\gamma_5\cdot\gamma_0\gamma_5\cdot\gamma_0\gamma_5\Phi_{-M,\pi}(\vec{\kappa}). \tag{21}$$

Really, by means of relation $\gamma_0\gamma_5 = -\gamma_5\gamma_0$ and direct evaluations we shall obtain:

$$\mathscr{P}\,\Phi'_M(\vec{\kappa}) = -\pi\Phi'_M(\vec{\kappa}). \tag{22}$$

Thus, the solution (21) describes an antibaryon. Taking into account the zero anticommutator (20), from the relation (21) the following relations imply also:

$$\left.\begin{aligned} \Phi^{+++}_{M,-\pi}(\vec{\kappa}) &= \gamma_0\gamma_5\cdot\gamma_0\gamma_5\cdot\gamma_0\gamma_5\Phi^{---}_{-M,\pi}(\vec{\kappa}), \\ \Phi^{---}_{M,-\pi}(\vec{\kappa}) &= \gamma_0\gamma_5\cdot\gamma_0\gamma_5\cdot\gamma_0\gamma_5\Phi^{+++}_{-M,\pi}(\vec{\kappa}), \end{aligned}\right\} \tag{23}$$

where the summands of positive and negative energy ($\Phi^{+++}_M(\vec{\kappa})$ and $\Phi^{---}_M(\vec{\kappa})$) of the total Salpeter amplitude

$$\Phi_M(\vec{\kappa}) = \Phi^{+++}_M(\vec{\kappa}) + \Phi^{---}_M(\vec{\kappa}) \tag{24}$$

are definited according to relations (9), (8).

At the same time the component-wise structure of Salpeter amplitude $\Phi_M(\vec{\kappa})$ and relation (21) (with an explicit form of matrix $\gamma_0\gamma_5\cdot\gamma_0\gamma_5\cdot\gamma_0\gamma_5$) allow to explain the physical sense of 8-component spinors in the meaning of "particle-antiparticle":

$$\left.\begin{array}{l}\Phi'_{M1,-\pi}(\vec{\kappa}) = -\Phi_{-M8,\pi}(\vec{\kappa})\,,\\ \Phi'_{M8,-\pi}(\vec{\kappa}) = \;\;\Phi_{-M1,\pi}(\vec{\kappa})\,.\end{array}\right\} \tag{25}$$

The baryon consisting of three constituent quarks is described by the total Salpeter amplitude $\Phi_M^{J^\pi M_J TM_T \mathscr{S}^*}$ having a quite definite quantum number set: JM_J – the total angular momentum and its projection, π – space parity, TM_T – the total isospin and its projection, $\mathscr{S}^*$ – strangeness. It is possible to become convinced that two 8-component spinors $\Phi_{M1,\pi}^{JM_J TM_T \mathscr{S}^*}$ and $\Phi_{M8,-\pi}^{JM_J TM_T \mathscr{S}^*}$ (which are components of the total amplitude) have almost the same quantum number set. Elimination is orbital parity. Here in the lower rows of indexes «π» and «$-\pi$» designate numerical values of orbital parity π'. In the former case numerically it is equal to the total space parity, and in the latter case it is opposite to it. All can be checked up acting with a parity operator on $\Phi_M(\vec{\kappa})$ (using an explicit form of matrix $\gamma_0\cdot\gamma_0\cdot\gamma_0$) and then line by line comparing the obtained expression with $\pi\cdot\Phi_M(\vec{\kappa})$.

Further we shall construct the required totally antisymmetric 8-component spinor $\Phi_{M1,\pi}^{JM_J TM_T \mathscr{S}^*}$ ($\Phi_{M8,-\pi}^{JM_J TM_T \mathscr{S}^*}$ can be constructed similarly; labels are borrowed from paper [8]):

$$\Phi_{M1,\pi}^{JM_J TM_T \mathscr{S}}(\vec{\kappa}) = \sum_{\mathscr{R}_L \mathscr{R}_S \mathscr{R}_F} \left\{\left\{\left\{\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathscr{R}_L}\cdot\left[\chi^S\right]_{\mathscr{R}_S}\right\}^{JM_J}\times\left[\varphi^{TM_T\mathscr{S}^*}\right]_{\mathscr{R}_F}\right\}_{\mathscr{S}}\cdot c_{\mathscr{A}}\right\}_{\mathscr{A}}. \tag{26}$$

Here $\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathscr{R}_L}$ is space wave function in $\vec{\kappa}$ -representation with total orbital momentum L, orbital parity π (coinciding with baryon total parity) and symmetry $\mathscr{R}_L \in \left\{\mathscr{S},\, \mathscr{M}_{\mathscr{S}}, \mathscr{M}_{\mathscr{A}}\right\}$ concerning permutations of quarks [22, 23]. $\left[\chi^S\right]_{\mathscr{R}_S}$ is 8-component (!) spin wave function (spinor) with total spin S and symmetry $\mathscr{R}_S \in \left\{\mathscr{S}, \mathscr{M}_{\mathscr{S}}, \mathscr{M}_{\mathscr{A}}\right\}$. $\left[\varphi^{TM_T\mathscr{S}^*}\right]_{\mathscr{R}_F}$ is flavour wave function with total isospin T and its projection M_T, with strangeness $\mathscr{S}^*$ and symmetry $\mathscr{R}_F \in \left\{\mathscr{S}, \mathscr{M}_{\mathscr{S}}, \mathscr{M}_{\mathscr{A}}, \mathscr{A}\right\}$. $c_{\mathscr{A}}$ is a wave function describing a totally antisymmetric colour (colourless) singlet.

Let us pay attention to one circumstance. In the proposed approach there is no need for the so-called relativization of Salpeter amplitude $\Phi_M^{J^\pi M_J TM_T \mathscr{S}^*}(\vec{\kappa})$ [8] as its spinor (relativistic) structure is ensured with required spinors $\Phi_{M1,\pi}^{JM_J TM_T \mathscr{S}^*}(\vec{\kappa})$ and $\Phi_{M8,-\pi}^{JM_J TM_T \mathscr{S}^*}(\vec{\kappa})$, whose 8-component structure in turn is set by spin wave function $\left[\chi^S\right]_{\mathscr{R}_S}$. All "remaining" gives us a solution of Salpeter equation with required unknown quantities in space wave function $\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathscr{R}_L}$.

The most labour-consuming is construction of a space wave function with a given symmetry. It is obtained by means of the so-called Young's symmetrization operators [9] comprising quark permutations. We have HH $\phi_K^{l_{q_j} l_{p_j} LM_L}(\Omega_{\vec{\kappa}})$ depending on Jacobi coordinate set (2). They satisfy the equation:

$$\hat{K}^2\phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}}) = K(K+4)\phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}}),\ \ K=0,1,2,..., \tag{27}$$

$$\hat{K}^2 = -\partial^2/\partial\alpha^2 - 4ctg2\alpha(\partial/\partial\alpha) + \hat{l}_q^2/\cos^2\alpha + \hat{l}_p^2/\sin^2\alpha. \tag{28}$$

$\hat{K}^2$ is square of 6-dimensional orbital momentum $\hat{K}$ or an angular part of 6-dimensional Laplacian:

$$\Delta_{\vec{\kappa}} \equiv \partial^2/\partial\vec{\kappa}^2 = \partial^2/\partial\kappa^2 + (5/\kappa)(\partial/\partial\kappa) - \hat{K}^2/\kappa^2\ . \tag{29}$$

HH are orthonormal functions:

$$\int\left(\phi_{K'}^{l'_q l'_p m'_q m'_p}(\Omega_{\vec{\kappa}})\right)^*\phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}})d\Omega_{\vec{\kappa}} = \delta_{K'K}\delta_{l'_q l_q}\delta_{l'_p l_p}\delta_{m'_q m_q}\delta_{m'_p m_p}\ . \tag{30}$$

Concrete expressions of HH can be found in [9]. However Young's symmetrization operators action on HH leads to need of having relation between HH depending on different Jacobi coordinate set obtained from starting set 1≡(123) by cyclical permutations of particles: 2≡(231), 3≡(312). Such relation is ensured with Reynal-Revai coefficients [24, 9]:

$$\phi_K^{l_{q_j} l_{p_j} L M_L}(\Omega_j) = \sum_{l_{q_i} l_{p_i}}\left\langle l_{q_i} l_{p_i}\middle| l_{q_j} l_{p_j}\right\rangle_{KL}\phi_K^{l_{q_i} l_{p_i} L M_L}(\Omega_i). \tag{31}$$

Because of invariancy of $\Delta_{\vec{\kappa}}$ concerning permutations of particles and rotations to Reynal-Revai coefficients conserved quantum numbers K and L are assigned. By means of relation (31) it is possible to express orbital wave function $\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathscr{R}_L}$ through a set of one type basis HH that considerably simplifies calculations. However per se HH $\phi_{KL}^{l_1 l_2}$ have no certain symmetry with respect to permutations group S_3, i.e. they are not basis functions of representation of this group. Thus there is a problem of constructing from $\phi_{KL}^{l_1 l_2}$ such complete and orthonormal basis $\phi_{KL}^{\{f\}\mu\sigma}$ which would be the basis of representation S_3. Such problem is solved by calculation of so-called symmetrization coefficients [9]. Here $\{f\}$ is Young's diagram: $\{3\}$ – symmetrical and $\{1^3\}$ – antisymmetric 1-dimensional representations, $\{21\}$ – 2-dimensional mixed symmetry representations; μ – Yamanouchi symbol for $\{21\}$ diagram; σ – the number of identical representations of S_3 groups.

For the elementary central three-partical interaction

$$\left.\begin{aligned}&\langle\vec{\kappa}|V|\vec{\kappa}'\rangle = \hat{\Pi}_0\mathscr{V}(\vec{\kappa},\vec{\kappa}')\ ,\\ &\hat{\Pi}_0 = \lim_{\vec{\kappa}\to 0}\hat{\Pi}(\vec{\kappa}) = \{1\cdot 1\cdot 1 + \gamma_0\cdot\gamma_0\cdot 1 + \gamma_0\cdot 1\cdot\gamma_0 + 1\cdot\gamma_0\cdot\gamma_0\}/4.\end{aligned}\right\} \tag{32}$$

Salpeter equation (1) has the following form:

$$\left.\begin{aligned}&\{M - \Omega B/A\}\Phi_1(\vec{\kappa}) - [\Omega\hat{C}/A]\Phi_8(\vec{\kappa}) = \Pi(\vec{\kappa})\int\mathscr{V}(\vec{\kappa},\vec{\kappa}')\Phi_1(\vec{\kappa}')[d\vec{\kappa}'/(2\pi)^6],\\ &[\Omega\hat{C}/A]\Phi_1(\vec{\kappa}) - \{M + \Omega B/A\}\Phi_8(\vec{\kappa}) = \Pi(\vec{\kappa})\int\mathscr{V}(\vec{\kappa},\vec{\kappa}')\Phi_8(\vec{\kappa}')[d\vec{\kappa}'/(2\pi)^6].\end{aligned}\right\} \tag{33}$$

The scalars are defined as follows:

$$\left.\begin{aligned}&\Omega(\vec{\kappa})=\omega_1+\omega_2+\omega_3,\\&A(\vec{\kappa})=m_1m_2\omega_3+m_1\omega_2m_3+\omega_1m_2m_3+\omega_1\omega_2\omega_3,\\&B(\vec{\kappa})=\omega_1\omega_2m_3+\omega_1m_2\omega_3+m_1\omega_2\omega_3+m_1m_2m_3,\\&\Pi(\vec{\kappa})=A(\vec{\kappa})/(4\omega_1\omega_2\omega_3)=1/(I^{1/4}\mathscr{N}(\vec{\kappa})).\end{aligned}\right\}\tag{34}$$

A single matrix $\hat{C}$ (which defines the spinor character of a set of equations (33)) has the form:

$$\hat{C}(\vec{\kappa})\equiv(\vec{\sigma}_1\cdot\vec{k}_1)(\vec{\sigma}_2\cdot\vec{k}_2)(\vec{\sigma}_3\cdot\vec{k}_3)\,,\tag{35}$$

where $\vec{\sigma}\equiv(\sigma_x,\sigma_y,\sigma_z)$ is a vector with components in the form of usual Pauli's matrixes. For variables A, B and $\hat{C}$ the relation $A^2=B^2+\hat{C}^2$ is valid.

The most popular (local) potentials used in calculations of two-quark bound systems [25] are easily generalized for the case of three-quark bound systems:

$$\langle\vec{\rho}|V_n|\vec{\rho}'\rangle=\hat{\Pi}_0\mathscr{V}_n(\vec{\rho},\vec{\rho}')=\hat{\Pi}_0\delta(\vec{\rho}-\vec{\rho}')[\mathscr{V}_0+\eta_n\rho^n e^{-\varepsilon\rho}],\tag{36}$$

where 6-vector $\vec{\rho}=(\vec{\eta},\vec{\xi})$ with length $\rho=\sqrt{\eta^2+\xi^2}$ (so-called collective variable) is made up of conjugate Jacobi coordinates

$$\left.\begin{aligned}&\vec{\eta}=\sqrt{\mu_1\mu_2/(\mu_1+\mu_2)}(\vec{x}_1-\vec{x}_2),\\&\vec{\xi}=-\sqrt{\mu_3(\mu_1+\mu_2)}[\vec{x}_3-(\mu_1\vec{x}_1+\mu_2\vec{x}_2)/(\mu_1+\mu_2)].\end{aligned}\right\}\tag{37}$$

Here $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ are usual Cartesian coordinates. $\vec{\kappa}$ - and $\vec{\rho}$ -vectors are costate variables. In relation (36) $\mathscr{V}_0$ and η_n are fitting parameters. Factor $e^{-\varepsilon}$ ($\varepsilon\to 0$) is introduced for regularization of potentials in $\vec{\kappa}$ -representation. Values $n=-1,1,2$ correspond to Coulomb, linear and oscillatory potentials respectively. Omitting fairly long evaluations we will write the expression $\mathscr{V}(\vec{\kappa},\vec{\kappa}')$ for the linear potential:

$$\mathscr{V}(\vec{\kappa},\vec{\kappa}')=(2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')\mathscr{V}_0+[(2\pi)^6/(\kappa\kappa')^2]\times\sum_K v_K^1(\kappa,\kappa';\varepsilon)\phi_K^*(\Omega_{\vec{\kappa}})\phi_K(\Omega_{\vec{\kappa}'}),\tag{38}$$

$$\begin{aligned}v_K^1(\kappa,\kappa';\varepsilon)=&(2K+3)/[\pi(\kappa\kappa')^{3/2}(z^2-1)]\cdot\{(z-z_0)[(K+5/2-2z^2/(z^2-1))\times\\&\times Q_{K+3/2}(z)+(2z/(z^2-1))Q_{K+1/2}(z)]+[zQ_{K+3/2}(z)-Q_{K+1/2}(z)]/2\},\end{aligned}\tag{39}$$

$$z=z_0+\varepsilon^2/(2\kappa\kappa'),\ \ z_0=(\kappa/\kappa'+\kappa'/\kappa)/2.\tag{40}$$

Here $Q_\nu(z)$ is Legendre function of the second kind.

However, oscillatory potential in momentum representation can be obtained in another more simple way. Fourier transform of potential (36) with $n=2$ and $\eta_2=M_0\Omega_0^2/2$ (Ω_0 - fitting parameter) in $\vec{\kappa}$ -representation has the form:

$$\begin{aligned}\mathscr{V}(\vec{\kappa},\vec{\kappa}')&=(2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')\mathscr{V}_0-(2\pi)^6[M_0\Omega_0^2/2]\left[\Delta_{\vec{\kappa}'}\delta(\vec{\kappa}-\vec{\kappa}')\right]=\\&=(2\pi)^6\delta(\vec{\kappa}-\vec{\kappa}')\left[\mathscr{V}_0-\vec{\kappa}^2/(2M_0)\right]+\sum_\nu\Omega_0(2N+K+3)\langle\nu|\vec{\kappa}'\rangle\langle\vec{\kappa}|\nu\rangle\,.\end{aligned}\tag{41}$$

In deriving relation (41) we used a completeness condition

$$\sum_{\nu} \langle \vec{\kappa} | \nu \rangle \langle \nu | \vec{\kappa}' \rangle = (2\pi)^6 \delta(\vec{\kappa} - \vec{\kappa}') \tag{42}$$

for 6-dimensional oscillatory basis functions $\langle \vec{\kappa} | \nu \rangle$ with quantum numbers $\nu \equiv {}^{l_q l_p L M_L}_{NK}$.
These functions satisfy the known oscillatory equation

$$\{-[M_0 \Omega_0^2 / 2]\Delta_{\vec{\kappa}} + \vec{\kappa}^2 / (2M_0) - (E - \mathscr{V}_0)\} \langle \vec{\kappa} | \Psi \rangle = 0 \tag{43}$$

and have the following form:

$$\left.\begin{aligned} &\langle \vec{\kappa} | \nu \rangle = (2\pi)^3 a \left[\chi_{NK}(a\kappa) / \kappa^{5/2} \right] \phi_K^{l_q l_p L M_L}(\Omega_{\vec{\kappa}}), \\ &\chi_{NK}(x) = \sqrt{2\Gamma(N+1)/[a\Gamma(N+K+3)]}\; e^{-x^2/2} x^{K+5/2} L_N^{K+2}(x^2) \;, \quad a = 1/\sqrt{M_0 \Omega_0} \;. \end{aligned}\right\} \tag{44}$$

$L_N^{K+2}(x^2)$ is Laguerre polynomial, N is positive integer, the energy eigenvalue is equal to

$$E_{NK} = \mathscr{V}_0 + \Omega_0 (2N + K + 3) \,. \tag{45}$$

It is easy to verify that the set of equations (33) has correct solutions in the case of the free motion: $\mathscr{V}(\vec{\kappa}, \vec{\kappa}') = 0$, and also has the correct nonrelativistic limit.

In summary it should be noted that having reduced the solution of Salpeter equation for three-particle bound systems to determination of two 8-component spinors without additional "relativization" of wave function, effectively using three-particle interquark interaction (in quark confinement conditions) and a powerful method of expansions in terms of hyperspherical harmonics (the most suitable to description of compact bound systems), we hope to appreciably simplify calculations of the basic characteristics of baryons.

The authors are grateful to N. Shubitidze for useful consultations.

The designated project has been fulfilled by financial support of the Georgia National Science Foundation (Grant #GNSF/ST08/4-401). Any idea in this publication is possessed by the author and may not represent the opinion of the Georgia National Science Foundation itself.

ფიზიკა

ბმული qqq-სისტემების შესწავლა სოლპიტერის განტოლების საფუძველზე უბრალო მიდგომის ფარგლებში ჰიპერსფერული ჰარმონიკების მწკრივად გაშლის მეთოდის გამოყენებით

თ. ბაბუციძე*, თ. კოპალეიშვილი, ვ. სხირტლაძე***

* *ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის მაღალი ენერგიების ფიზიკის ინსტიტუტი*
** *აკადემიის წევრი, ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის მაღალი ენერგიების ფიზიკის ინსტიტუტი*

სოლპიტერის განტოლების საფუძველზე სამი ბმული კონსტიტუენტური კვარკის (ბარიონის) ამოცანის ამოხსნისათვის, სრული ტალღური ფუნქციის ე.წ. რელატივიზაციის გარეშე ყალიბდება

მიდგომა, სადაც ორ საძიებელ 8-კომპონენტიან სპინორს გააჩნია ნათელი ფიზიკური შინაარსი ნაწილაკ-ანტინაწილაკის (ბარიონ-ანტიბარიონის) კონტექსტში. მითითებულია ორნაწილაკოვანი ურთიერთქმედების განხილვის საეჭვო ხასიათი კვარკული კონფაინმენტის პირობებში. სამნაწილაკოვანი ურთიერთქმედების მქონე კომპაქტური ბმული სისტემების ყველაზე სრული და ადეკვატური აღწერისათვის შემოთავაზებულია გამოთვლებში ჰიპერსფერული ჰარმონიკების მწკრივად გაშლის მეთოდის გამოყენება. განხილულია ცენტრალური სამნაწილაკოვანი ურთიერთქმედების ორი მარტივი ტიპი – წრფივი და ოსცილატორული პოტენციალები.

REFERENCES

1. *S. Capstick, W. Roberts* (2000), Prog. Part. Nucl. Phys., **45:** 241-331.
2. *E. E. Salpeter and H. A. Bethe* (1951), Phys. Rev., **84**, 6: 1231-1242.
3. *J. G. Taylor* (1966), Phys. Rev., **150**, 4: 1321-1330.
4. *M. Gell-Mann, F. E. Low* (1951), Phys. Rev., **84**, 2: 350-354.
5. *S. Weinberg* (1995), The Quantum Theory of Fields, **1**, 10.2, Polology: 428. Cambridge University Press, New York.
6. *E. E. Salpeter* (1952), Phys. Rev., **87**, 2: 328-343.
7. *V.R. Garsevanishvili, A.N. Tavkhelidze* (1999), Phys. Part. Nucl., **30**, 3: 612-659.
8. *U. Loring et al.* (2001), Eur. Phys. J., **A**, **10**, 3: 309-346.
9. *R. I. Jibuti, N. B. Krupennikova* (1984), Metod gipersfericheskikh funktsiy v kvantovoy mekhanike neskol'kikh tel. Tbilisi (in Russian).
10. *James Avery, John Avery* (2006), Generalized Sturmians and Atomic Spectra, World Scientific. New Jersey.
11. *John Avery* (2002), Hyperspherical Harmonics and Generalized Sturmians. Kluwer Academic Publishers, New York.
12. *G. I. Kuznetsov et al.* (1985), Graficheskaya teoriya predstavleniy ortogonal'nykh i unitarnykh grupp i ee fizicheskie prilozheniya. Kiev (in Russian).
13. *A. Archvadze et al.* (1993), Few-Body Systems, **14**: 53-58.
14. *A. Archvadze et al.* (1985), Nucl.Phys. **A**, **581**: 460-470.
15. *T. Babutsidze et al.* (1998), Phys. Lett. **B**, **426**: 139.
16. *T. Babutsidze et al.* (1999), Phys. Rev. **C**, **59**: 976.
17. *T. Babutsidze et al.* (2004), GESJ Phys., **39**: 20.
18. *T. Kopaleishvili* (2001), Phys. Part. Nucl., **32**, 5: 1060-1114.
19. *A. A. Logunov and A. N. Tavkhelidze* (1963), Nuovo Cimento, **29**, 2: 380.
20. *R. P. Feynman* (1949), Phys. Rev., **76**, 6: 749-759, 769-789.
21. *Z. K. Silagadze* (2009), Wick-Katkovsky model: an introduction, hep-ph/9803307v3.
22. *F. E. Close* (1979), An Introduction to Quarks and Partons. Academic Press, London.
23. *F. Halzen, A. D. Martin* (1984), Quarks and Leptons. John Wiley&Sons, New York.
24. *J. Raynal and J. Revai* (1970), Nuovo Cimento **A**, **68**, 4: 612-622.
25. *D. Eyre and J. P. Vary* (1987), Phys. Rev. **D**, **34**, 11: 3467-3471.

Received December, 2011