Mechanics

Nonlocal Refined Theory for Nanobeams with Surface Effects

Isaac Elishakoff[*] and Clement Soret[**]

[*] *Department of Ocean and Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431-0991, USA Foreign Member of the Georgian National Academy of Sciences*
[**] *French Institute for Advanced Mechanics, Aubière, 63175, France*

ABSTRACT. In this study we propose governing differential equations for beams taking into account shear deformation, rotary inertia, locality and surface stress effects. It is shown that the equation is both simpler and more consistent than the appropriate Bresse-Timoshenko equations extended to include locality and surface stress effects. Proposed equation contains 11 terms with respect to displacement versus 19 terms appearing in the equations that extend the Bresse-Thimoshenko equations to include nonlocality and surface effects. © *2012 Bull. Georg. Natl. Acad. Sci.*

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1. Introduction.

We consider the following Bresse-Timoshenko equations for a beam analyzed by local theory and without surface effects

$$EI\frac{\partial^2\phi}{\partial x^2}-\kappa AG\left(\phi-\frac{\partial v}{\partial x}\right)-\rho I\frac{\partial^2\phi}{\partial t^2}=0, \tag{1}$$

$$\rho A\frac{\partial^2 v}{\partial t^2}+\kappa AG\left(\frac{\partial\phi}{\partial x}-\frac{\partial^2 v}{\partial x^2}\right)=0, \tag{2}$$

where κ is the shear coefficient which depends on the shape of the cross-section, A is the area of the cross-section, G is the modulus of elasticity in shear, ϕ is the slope of the deflection curve when the shear force is neglected, v is the total deflection, x is the axial displacement, ρ is the mass density per unit volume, t is the time, E is the Young's modulus of elasticity and I is the area moment of inertia.

Equations (1) and (2) can be reduced to a single differential equation in terms of displacement $v(x,t)$:

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 v}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4 v}{\partial t^4}=0. \tag{3}$$

Timoshenko [1] evaluated the natural frequencies of the simply supported beam at both ends and arrived at the conclusion that the term associated with the last term in equation (3) " is small quantity of the second order compared with" other terms in the characteristic equation.

Let us look at the third term in dynamic equilibrium equation considered by Timoshenko

$$-V+\frac{\partial M}{\partial x}-\rho I\frac{\partial^2\phi}{\partial t^2}=0 \tag{4}$$

namely $-\rho I\partial^2\phi/\partial t^2$ which replaces its counterpart $-\rho I\partial^3 v/\partial x\partial t^2$ in equation associated with taking into account the rotary inertia alone

$$-V+\frac{\partial M}{\partial x}-\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0\,. \tag{5}$$

Timoshenko's [1] purpose was, as the word "correction" in the title of his paper suggests, a correction of the original rotary inertia term by Bresse [2] and Rayleigh [3] by incorporating the shearing force. However, it appears that such a correction has a secondary effect, since it attempts to correct the correction due to the rotary inertia effect. Such a process of correction of correction apparently leads to secondary effect and results in an inconsistent theory including, as was shown by Timoshenko [1] himself, a second order term.

The situation was remedied by Elishakoff [4]. Namely, it was shown that simpler and more consistent set of equations is derivable by retaining the equation (2) whereas the equation (1) should be replaced by

$$EI\frac{\partial^2\phi}{\partial x^2}-\kappa AG\left(\phi-\frac{\partial v}{\partial x}\right)-\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0\,. \tag{6}$$

Equations (2) and (6) can also be reduced to the single differential equation

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 v}{\partial x^2\partial t^2}=0\,. \tag{7}$$

As is seen, the consistent equation (7) differs from the original Bresse-Timoshenko equation (3) by absence of the term containing the fourth derivative with respect to time.

It will be our goal to develop consistent equation also for the flexural beams within nonlocal theory with or without surface effects.

2. Analysis without surface effects.

Lu, Lee, Lu and Zhang [5] and Lu, Lee, Lu and Zhang [6] proposed a generalization of Bresse-Timoshenko equations for the flexural beams analyzed by the nonlocal theory without surface effects

$$EI\frac{\partial^2\phi}{\partial x^2}+\kappa GA\left(\frac{\partial v}{\partial x}-\phi\right)-\left[1-\left(e_0 a\right)^2\frac{\partial^2}{\partial x^2}\right]\rho I\frac{\partial^2\phi}{\partial t^2}=0\,, \tag{8}$$

$$\kappa GA\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\left[1-\left(e_0 a\right)^2\frac{\partial^2}{\partial x^2}\right]\rho A\frac{\partial^2 v}{\partial t^2}=-\left[p-\left(e_0 a\right)^2\frac{\partial^2 p}{\partial x^2}\right], \tag{9}$$

where e_0a is the scale coefficient that incorporates the nonlocality effect, a is an internal characteristic length, and e_0 is a constant for adjusting the model in matching some reliable results by experiments or other models. When $e_0a = 0$, equations (8) and (9) are reduced to the equations of classical Bresse-Timoshenko beam.

We can extract $\partial\phi/\partial x$ from equation (9) to get

$$\frac{\partial\phi}{\partial x}=\frac{\partial^2 v}{\partial x^2}-\frac{1}{\kappa GA}\left(\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho A\frac{\partial^2 v}{\partial t^2}-\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right]\right). \tag{10}$$

Differentiation of equation (8) with respect to x yields

$$EI\frac{\partial^3\phi}{\partial x^3}+\kappa GA\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\left[\frac{\partial}{\partial x}-(e_0a)^2\frac{\partial^3}{\partial x^3}\right]\rho I\frac{\partial^2\phi}{\partial t^2}=0. \tag{11}$$

In order to substitute $\partial^3\phi/\partial x^3$, $\partial^3\phi/\partial x\partial t^2$ and $\partial^5\phi/\partial x^3\partial t^2$ in equation (11), we first write the following derivatives using equation (10)

$$\frac{\partial^3\phi}{\partial x^3}=\frac{\partial^4 v}{\partial x^4}-\frac{1}{\kappa GA}\left(\left[\frac{\partial^2}{\partial x^2}-(e_0a)^2\frac{\partial^4}{\partial x^4}\right]\rho A\frac{\partial^2 v}{\partial t^2}-\left[\frac{\partial^2 p}{\partial x^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^4}\right]\right), \tag{12}$$

$$\frac{\partial^3\phi}{\partial x\,\partial t^2}=\frac{\partial^4 v}{\partial x^2\partial t^2}-\frac{1}{\kappa GA}\left(\left[\frac{\partial^2}{\partial t^2}-(e_0a)^2\frac{\partial^4}{\partial x^2\partial t^2}\right]\rho A\frac{\partial^2 v}{\partial t^2}-\left[\frac{\partial^2 p}{\partial t^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right]\right), \tag{13}$$

$$\frac{\partial^5\phi}{\partial x^3\partial t^2}=\frac{\partial^6 v}{\partial x^4\partial t^2}-\frac{1}{\kappa GA}\left(\left[\frac{\partial^4}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^2}{\partial x^4\partial t^2}\right]\rho A\frac{\partial^2 v}{\partial t^2}-\left[\frac{\partial^4 p}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right]\right). \tag{14}$$

Substituting equations (10), (12), (13) and (14) into equation (11), equations (8) and (9) are reduced to a single differential equation in terms of displacement $v(x,t)$:

$$\begin{aligned}
&EI\left(\frac{\partial^4 v}{\partial x^4}-\frac{\rho}{\kappa G}\left[\frac{\partial^4 v}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 v}{\partial x^4\partial t^2}\right]\right)+\rho A\left(\frac{\partial^2 v}{\partial t^2}-(e_0a)^2\frac{\partial^4 v}{\partial x^2\partial t^2}\right)-\\
&-\rho I\left(\frac{\partial^4 v}{\partial x^2\partial t^2}-\frac{\rho}{\kappa G}\left[\frac{\partial^4 v}{\partial t^4}-(e_0a)^2\frac{\partial^6 v}{\partial x^2\partial t^4}\right]\right)+\\
&+(e_0a)^2\rho I\left(\frac{\partial^6 v}{\partial x^4\partial t^2}-\frac{\rho}{\kappa G}\left[\frac{\partial^6 v}{\partial x^2\partial t^4}-(e_0a)^2\frac{\partial^8 v}{\partial x^4\partial t^4}\right]\right)=-\frac{EI}{\kappa GA}\left[\frac{\partial^2 p}{\partial x^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^4}\right]+\\
&+\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right]+\frac{1}{\kappa GA}\left[\frac{\partial^2 p}{\partial t^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right]+(e_0a)^2\rho I\left[\frac{\partial^4 p}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right].
\end{aligned} \tag{15}$$

We would like to compare the equation (15) with the equation that is derivable via consistent analysis. Following Ref. [4] a simpler and more consistent set of equations is constituted by retaining the equation (9) whereas the equation (8) should be replaced by

$$EI\frac{\partial^2\phi}{\partial x^2}+\kappa GA\left(\frac{\partial v}{\partial x}-\phi\right)-\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0. \qquad (16)$$

Then equation (10) should be retained. Differentiation of equation (16) with respect to x yields

$$EI\frac{\partial^3\phi}{\partial x^3}+\kappa GA\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\left[\frac{\partial}{\partial x}-(e_0a)^2\frac{\partial^3}{\partial x^3}\right]\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0. \qquad (17)$$

Substituting equations (10) and (12) into equation (17), equations (9) and (16) we obtain a single differential equation in terms of displacement $v(x,t)$:

$$EI\left(\frac{\partial^4 v}{\partial x^4}-\frac{1}{\kappa GA}\left[\frac{\partial^2}{\partial x^2}\left(\rho A\frac{\partial^2 v}{\partial t^2}\right)-(e_0a)^2\frac{\partial^4}{\partial x^4}\left(\rho A\frac{\partial^2 v}{\partial t^2}\right)\right]\right)+\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho A\frac{\partial^2 v}{\partial t^2}-$$

$$-\rho I\left[\frac{\partial^4 v}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 v}{\partial x^4\partial t^2}\right]=-\frac{EI}{\kappa GA}\left[\frac{\partial^2 p}{\partial x^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^4}\right]+\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right]. \qquad (18)$$

As is seen, the consistent set differs from the original Bresse-Timoshenko equations by absence of the following terms

$$-\frac{\rho A}{\kappa GA}\left[\frac{\partial^4 v}{\partial t^4}-(e_0a)^2\frac{\partial^6 v}{\partial x^2\partial t^4}\right]\rho I-\frac{(e_0a)^2\rho^2 I}{\kappa G}\left[\frac{\partial^6 v}{\partial x^2\partial t^4}-(e_0a)^2\frac{\partial^8 v}{\partial x^4\partial t^4}\right]$$

$$\frac{1}{\kappa GA}\left[\frac{\partial^2 p}{\partial t^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right](e_0a)^2\rho I\left[\frac{\partial^4 p}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right]. \qquad (19)$$

Specifically, equation (15) consists of 11 terms containing $v(x,t)$ whereas equation (18) consists of 7 such terms. Moreover, equation (15) contains 8 terms containing $p(x,t)$, whereas equation (18) contains only 4 such terms.

As expected, the consistent set differs from the original Bresse-Timoshenko equations by absence of the terms containing the fourth derivative with respect to time. Moreover, four additional terms with respect to $v(x,t)$, and four additional terms with respect to loading $p(x,t)$ are absent in the consistent set.

3. Analysis with surface effects

Lee and Chang [7] derived the equations of motion for the nonlocal Bresse-Timoshenko beam model with surface effects as follows

$$(EI)^*\frac{\partial^2\phi}{\partial x^2}+\kappa GA\left(\frac{\partial v}{\partial x}-\phi\right)-\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho I\frac{\partial^2\phi}{\partial t^2}=0, \qquad (20)$$

$$\kappa GA\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]=-\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right], \qquad (21)$$

where $(EI)^*$ is the effective flexural rigidity which includes the surface bending elasticity on the nanotube and its flexural rigidity and defined as $(EI)^* = \pi E^S \left(R_i^3 + R_o^3\right) + EI$, with E^S being the surface elasticity modulus, H is the constant parameter which is determined by the residual surface tension and the shape of cross section and defined as $H = 4\tau\left(R_i + R_o\right)$, with τ the residual surface tension per length on the nanotube.

We extract $\partial\phi/\partial x$ from equation (21) to get

$$\frac{\partial\phi}{\partial x} = \frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa GA}\left(\left[1-(e_0 a)^2\frac{\partial^2}{\partial x^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right] - \left[p - (e_0 a)^2\frac{\partial^2 p}{\partial x^2}\right]\right). \tag{22}$$

Differentiation of equation (20) with respect to x yields

$$(EI)^*\frac{\partial^3\phi}{\partial x^3} + \kappa GA\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial\phi}{\partial x}\right) - \left[\frac{\partial}{\partial x} - (e_0 a)^2\frac{\partial^3}{\partial x^3}\right]\rho I\frac{\partial^2\phi}{\partial t^2} = 0\,. \tag{23}$$

In order to substitute $\partial^3\phi/\partial x^3$, $\partial^3\phi/\partial x\partial t^2$ and $\partial^5\phi/\partial x^3\partial t^2$ in equation (23), we obtain the following derivatives using equation (22)

$$\frac{\partial^3\phi}{\partial x^3} = \frac{\partial^4 v}{\partial x^4} - \frac{1}{\kappa GA}\left(\left[\frac{\partial^2}{\partial x^2}-(e_0 a)^2\frac{\partial^4}{\partial x^4}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right] - \left[\frac{\partial^2 p}{\partial x^2} - (e_0 a)^2\frac{\partial^4 p}{\partial x^4}\right]\right), \tag{24}$$

$$\frac{\partial^3\phi}{\partial x\,\partial t^2} = \frac{\partial^4 v}{\partial x^2\partial t^2} - \frac{1}{\kappa GA}\left(\left[\frac{\partial^2}{\partial t^2}-(e_0 a)^2\frac{\partial^4}{\partial x^2\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right] - \left[\frac{\partial^2 p}{\partial t^2} - (e_0 a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right]\right), \tag{25}$$

$$\frac{\partial^5\phi}{\partial x^3\partial t^2} = \frac{\partial^6 v}{\partial x^4\partial t^2} - \frac{1}{\kappa GA}\left(\left[\frac{\partial^4}{\partial x^2\partial t^2}-(e_0 a)^2\frac{\partial^6}{\partial x^4\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right] - \left[\frac{\partial^4 p}{\partial x^2\partial t^2} - (e_0 a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right]\right). \tag{26}$$

Substituting equations (22), (24), (25) and (26) into equation (23), equations (20) and (21) can be reduced to a single equation:

$$\begin{aligned}
&(EI)^*\left(\frac{\partial^4 v}{\partial x^4} - \frac{1}{\kappa GA}\left[\frac{\partial^2}{\partial x^2}-(e_0 a)^2\frac{\partial^4}{\partial x^4}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]\right) + \left[1-(e_0 a)^2\frac{\partial^2}{\partial x^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right] \\
&\quad -\rho I\left\{\frac{\partial^4 v}{\partial x^2\partial t^2} - \frac{1}{\kappa GA}\left[\frac{\partial^2}{\partial t^2}-(e_0 a)^2\frac{\partial^4}{\partial x^2\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]\right. \\
&\quad \left. -(e_0 a)^2\left(\frac{\partial^6 v}{\partial x^4\partial t^2} - \frac{1}{\kappa GA}\left[\frac{\partial^4}{\partial x^2\partial t^2}-(e_0 a)^2\frac{\partial^6}{\partial x^4\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]\right)\right\} \\
&\quad = -\frac{(EI)^*}{\kappa GA}\left[\frac{\partial^2 p}{\partial x^2} - (e_0 a)^2\frac{\partial^4 p}{\partial x^4}\right] + \left[p - (e_0 a)^2\frac{\partial^2 p}{\partial x^2}\right] + \frac{\rho I}{\kappa GA}\left\{\left[\frac{\partial^2 p}{\partial t^2} - (e_0 a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right]\right. \\
&\quad \left. -(e_0 a)^2\left[\frac{\partial^4 p}{\partial x^2\partial t^2} - (e_0 a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right]\right\}.
\end{aligned} \tag{27}$$

A simpler and more consistent set of equations is constituted (see Ref. [4]) by retaining the equation (21) whereas the equation (20) should be replaced by

$$(EI)^*\frac{\partial^2\phi}{\partial x^2}+\kappa GA\left(\frac{\partial v}{\partial x}-\phi\right)-\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0. \qquad (28)$$

Then equation (22) remains and differentiation with respect to x yields

$$(EI)^*\frac{\partial^3\phi}{\partial x^3}+\kappa GA\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\left[\frac{\partial}{\partial x}-(e_0a)^2\frac{\partial^3}{\partial x^3}\right]\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0 \qquad (29)$$

and substituting equations (23) and (24) into equation (29) gives

$$(EI)^*\left(\frac{\partial^4 v}{\partial x^4}-\frac{1}{\kappa GA}\left[\frac{\partial^2}{\partial x^2}-(e_0a)^2\frac{\partial^4}{\partial x^4}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]\right)+\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]$$

$$-\left[\frac{\partial}{\partial x}-(e_0a)^2\frac{\partial^3}{\partial x^3}\right]\rho I\frac{\partial^3 v}{\partial x\partial t^2}=-\frac{(EI)^*}{\kappa GA}\left[\frac{\partial^2 p}{\partial x^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^4}\right]+\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right]. \qquad (30)$$

As is seen, the consistent set differs from the original Lee and Chang [7] equations by absence of the following terms

$$\frac{\rho I}{\kappa GA}\left[\frac{\partial^2}{\partial t^2}-(e_0a)^2\frac{\partial^4}{\partial x^2\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]$$

$$\frac{1}{\kappa GA}(e_0a)^2\left[\frac{\partial^4}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6}{\partial x^4\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]$$

$$\frac{\rho I}{\kappa GA}\left[\frac{\partial^2 p}{\partial t^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right]-\frac{\rho I}{\kappa GA}(e_0a)^2\left[\frac{\partial^4 p}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right]. \qquad (31)$$

Note that equation (27) corresponding to the original Bresse-Timoshenko theory with nonlocality and surface effects incorporates 19 terms for displacement $v(x,t)$ whereas equation (30) contains 11 such terms. Likewise, equation (27) consists of 8 terms for the distributed load $p(x,t)$ whereas equation (30) contains 4 such terms.

As observed in the case without surface effects, comparison of the consistent set with the original Bresse-Timoshenko equations shows the absence of the terms containing the fourth derivative with respect to time and eight additional terms.

4. Free Vibrations of Nonlocal Bernoulli-Euler Beams

We study free vibrations of the Bresse-Timoshenko beams that are simply-supported at both ends. To do so, we postulate that harmonic vibrations with natural frequency ω are taking place. The boundary conditions are satisfied if we let

$$v(x,t)=V\sin\frac{m\pi x}{L}\sin\omega t, \qquad (32)$$

where m is the number of half-waves in axial direction, L is the beam length. In equations (15), (18), (27) and (30) we let $p(x,t) \equiv 0$.

We would like to compare the solutions obtained for ω^2 from equations (15), (18), (27) and (30) with the solution derived from the nonlocal Bernoulli-Euler beam, namely

$$\omega_{BE}^2 = \frac{EI}{\rho A\left[1+\left(e_0 a\right)^2\left(\frac{m\pi}{L}\right)^2\right]}\left(\frac{m\pi}{L}\right)^4. \tag{33}$$

This equation corresponds to the equation discussed by Zhang, Liu and Xie [8] and Lu, Lee, Lu and Zhang [9]. As is seen the squared natural frequency of the nonlocal Bernoulli-Euler beam is smaller than its counterpart for the beam treated by local theory $EI\left(m\pi/L\right)^4/\rho A$.

5. Free Vibration Analysis of Bresse-Timoshenko Beams without Surface Effects

Substitution of equation (32) obtained from the original set into equation (15) leads to the following expression for ω_O^2

$$\begin{aligned}\omega_O^2 = \frac{1}{2}\frac{1}{\rho I\left(L^4+L^2\left(e_0 a\right)^2\pi^2 m^2+\left(e_0 a\right)^4\pi^4 m^4\right)}\Big[\Big(&EIL^2 m^2\pi^2 + EIm^4\pi^4\left(e_0 a\right)^2 + A\kappa GL^4 \\ &+A\kappa GL^2\left(e_0 a\right)^2\pi^2 m^2 + I\pi^2 m^2\kappa GL^2 + I\left(e_0 a\right)^2\pi^4 m^4\kappa G - \Big(A^2\kappa^2 G^2 L^8 - 2EI^2 L^4\pi^4 m^4\kappa G \\ &-2EI^2\pi^8 m^8\left(e_0 a\right)^2\kappa G + 2A^2\kappa^2 G^2 L^6\left(e_0 a\right)^2\pi^2 m^2 + 2AI\kappa^2 G^2 L^6\pi^2 m^2 + A^2\kappa^2 G^2 L^4\left(e_0 a\right)^2\pi^4 m^4 \\ &+2I^2 m^6\pi^6\kappa^2 G^2 L^2\left(e_0 a\right)^2 + E^2 I^2 L^4 m^4\pi^4 + E^2 I^2 m^8\pi^8\left(e_0 a\right)^4 + 2E^2 I^2 L^2 m^6\pi^6\left(e_0 a\right)^2 \\ &+I^2 L^4 m^4\pi^4\kappa^2 G^2 + I^2\left(e_0 a\right)^4 m^8\pi^8\kappa^2 G^2 + 2EIL^6 A\kappa Gm^2\pi^2 + 4EIL^4 A\kappa Gm^4\pi^4\left(e_0 a\right)^2 \\ &+2EIA\kappa GL^2 m^6\pi^6\left(e_0 a\right)^4 + 4IL^4 A\kappa^2 G^2 m^4\pi^4 + 2AI\kappa^2 G^2 L^2 m^6\pi^6\left(e_0 a\right)^4\Big)^{1/2}\Big].\end{aligned} \tag{34}$$

Equation (18) associated with the consistent set yields the following equation for squared natural frequency

$$\omega_C^2 = \frac{EI\pi^4 m^4\kappa G}{\rho\left(EI\pi^2 m^2 L^2 + EI\pi^4 m^4\left(e_0 a\right)^2 + A\kappa GL^4 + A\kappa GL^2 m^2\pi^2\left(e_0 a\right)^2 + I\pi^2 m^2\kappa GL^2 + I\pi^4 m^4\kappa G\left(e_0 a\right)^2\right)}. \tag{35}$$

In order to compare the natural frequencies obtained by original and consistent set we form the ratio ω_C^2/ω_O^2. The first natural frequencies are $\omega_O^{(1)} = 1.293109212\cdot 10^{12}$ rad, $\omega_C^{(1)} = 1.285469485\cdot 10^{12}$ rad, respectively. As is seen the difference is less than 0.60%. The second frequencies equal $\omega_O^{(2)} = 3.702494824\cdot 10^{12}$ rad and $\omega_C^{(2)} = 3.600545631\cdot 10^{12}$ rad, respectively, the difference being less than 2.76%. For the third frequency ($\omega_O^{(3)} = 5.891090015\cdot 10^{12}$ rad, $\omega_C^{(3)} = 5.636053802\cdot 10^{12}$ rad) the difference constitutes less than 4.33%.

As is seen, the consistent set yields nearly the same fundamental natural frequency as the original set, but the equation is much simpler than the one associated with original Bresse-Timoshenko theory. Moreover, as shown in Ref. 4, the simpler set is also more consistent than the original one.

6. Free Vibrations Analysis of Bresse-Timoshenko Beams with Surface Effects

Substitution of equation (32) obtained from the original set into equation (27) leads to the following expression for ω_O^2

$$
\begin{aligned}
\omega_O^2 = \frac{1}{2\rho IAL^2\left(L^2+\left(e_0a\right)^2\pi^2m^2\right)}\Big[& A^2\kappa GL^4 + L^2IH\pi^2m^2 + L^2\left(EI\right)^* A\pi^2m^2 + L^2I\kappa GA\pi^2m^2 \\
& + IH\left(e_0a\right)^2 m^2\pi^2 - \Big(2L^2I^2H^2m^6\pi^6\left(e_0a\right)^2 + L^4I^2H^2\pi^4m^4 + L^4\left(EI\right)^{*2}\pi^4m^4A^2 \\
& + I^2H^2\left(e_0a\right)^4 m^8\pi^8 + 2A^3\kappa GL^6\left(EI\right)^*\pi^2m^2 + 2A^3\kappa^2G^2L^6I\pi^2m^2 - 2L^4IH\pi^4m^4\left(EI\right)^* A \\
& + L^4\left(EI\right)^{*2}\pi^4m^4G^2A^2 + A^4\kappa^2G^2L^2 - 2A^2\kappa GL^6IH\pi^2m^2 - 2A\kappa GL^4IH\left(e_0a\right)^2\pi^4m^4 \\
& + 2L^4I^2H\kappa GA\pi^4m^4 - 2L^4\left(EI\right)^* A^2I\kappa G + 2L^2\left(EI\right)^*\pi^6m^6AIH\left(e_0a\right)^2 \\
& + 2L^2I^2\pi^6m^6\kappa GAH\left(e_0a\right)^2 + 4IA^2L^2\left(EI\right)^*\pi^6m^6\left(e_0a\right)^4\rho H\Big)^{1/2}\Big].
\end{aligned}
\tag{36}
$$

Equation (30) obtained from the consistent set yields

$$
\omega_C^2 = \frac{\pi^2m^2\left[\left(EI\right)^*\pi^2m^2L^2\kappa GA + \left(EI\right)^*\pi^2m^2HL^2 + \left(EI\right)^*H\pi^4m^4\left(e_0a\right)^2 + HL^4\kappa GA + \left(e_0a\right)^2H\pi^2m^2L^2\kappa GA\right]}{\rho AL^2\left[\left(EI\right)^*\pi^2m^2L^2 + \left(EI\right)^*\pi^4m^4\left(e_0a\right)^2 + AL^4\kappa G + \left(e_0a\right)^2A\pi^2m^2L^2\kappa G + I\pi^2m^2L^2\kappa G + I\pi^4m^4\kappa G\left(e_0a\right)^2\right]}. \tag{37}
$$

The first natural frequencies are $\omega_O^{(1)} = 1.951736705\cdot 10^{12}$ rad and $\omega_C^{(1)} = 1.941921730\cdot 10^{12}$ rad, respectively. As is seen, the difference is less than 0.51%. The second frequencies equal, respectively, $\omega_O^{(2)} = 4.726408202\cdot 10^{12}$ rad and $\omega_C^{(2)} = 4.644847128\cdot 10^{12}$ rad, difference being 1.76%. For the third frequency ($\omega_O^{(3)} = 7.116491920\cdot 10^{12}$ rad, $\omega_C^{(3)} = 7.009604822\cdot 10^{12}$ rad) the percentagewise difference constitutes 1.53%.

As is seen, the consistent set yields nearly the same fundamental natural frequency as the original set, but the equation is much simpler than the one associated with original Bresse-Timoshenko theory. Moreover, as shown in Ref. [4], the simpler set is also more consistent than the original one.

7. Conclusion

In this paper we propose a consistent set of nonlocal Bresse-Timoshenko equations for nonlocal beams with or without surface stress effects. We demonstrate that for the lower end of natural frequencies the consistent set and original set yield close values for natural frequencies. Moreover, the suggested governing equation is simpler than that associated with original equations extended to include nonlocal and surface effects.

Our conclusions are compatible with the results of study by Sadeghian *et al* [10] who conclude that "surface stress can cause significant difference in resonant frequency compared to the value obtained by classical beam theory."

It is hoped that equations developed in this study will find wide application in engineering research and practice.

მექანიკა

ნანოღეროების არალოკალური დაზუსტებული თეორია ზედაპირული ეფექტებით

ი. ელიშაკოვი[*] და კ. სორე[**]

* *ატლანტიკის უნივერსიტეტი, ოკეანისა და მექანიკის დეპარტამენტი, ფლორიდა, აშშ*
საქართველოს მეცნიერებათა ეროვნული აკადემიის უცხოელი წევრი

** *საფრანგეთის უმაღლესი მექანიკის ინსტიტუტი, ობიერი, საფრანგეთი*

მოცემულ გამოკვლევაში შემოთავაზებულია დიფერენციალური განტოლებები ღეროებისათვის რომლებიც ითვალისწინებენ განივ დეფორმაციებს, ბრუნვის ინერციას, ლოკალურობას და ზედაპირული ძაბვის გავლენას. ნაჩვენებია, რომ ჩვენს მიერ შემოთავაზებული განტოლება არის უფრო მარტივიც და თავსებადიც ვიდრე შესაბამისი ბრესსე-ტიმოშენკოს განტოლებები და მასში ჩართულია ლოკალურობის და ზედაპირული ძაბვის ეფექტი. შემოთავაზებული განტოლება შეიცავს 11 წევრს გადაადგილების ვექტორის მიმართ ნაცვლად 19 წევრისა, რომლებიც მონაწილეობენ ბრესსე-ტიმოშენკოს განტოლებების პირდაპირ განზოგადებაში, რაც ითვალისწინებს არალოკალურობას და ზედაპირულ ეფექტებს.

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