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Mathematics

Partially Independent Random Variables

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ABSTRACT. In this paper the definition of A- independence of X and Y random variables is introduced and the example of A-independent random variables is constructed. Regression of X on Y and regression of Y on X are investigated. Also the joint characteristic function of this random variables is obtained.

Key words: *random variables, A-independence,regression, characteristic function.*

Introduction. One of the important and fundamental problems of probability theory is independence of random variables. In this paper we introduce the definition of partial independence of two random variables. Using the standard bivariate normal distribution density we construct a nontrivial example of joint probability distribution density for such partially independent random variables.We investigate the properties of this distribution, find the conditional probability distribution density and calculated regressions. Also we give the expression for characteristic function of this joint probability distribution.

1. A-independent random variables.

Definition. We say that real random variables X and Y on the probability space (Ω, F, P) are A-independent (A is the subset of R^2) if and only if $F_{XY}(x,y) = F_x(x)F_Y(y)$ for all $(x,y) \in A$, where $F_{XY}(x,y) = P(X \le x, Y \le y)$ is the joint probability distribution function of X and Y, $F_X(x)$ and $F_Y(y)$ are the probability distribution functions of X and Y respectively.

It is clear that independence in the usual sense (see definition, for examples, in [1]) of random variables X and Y coincides with $A = R^2$ -independence.

If there exists the joint probability distribution density $f_{XY}(x,y)$, then we say that X and Y are A-independent if and only if $f_{XY}(x,y) = f_X(x)f_y(y)$ for all $(x,y) \in A$, where $f_X(x)$ and $f_y(y)$ are the probability distribution densities of X and Y respectively.

We begin to construct a special example of A-independent random variables using the joint standard normal distribution density

$$f(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\{-\frac{(x^2+y^2-2xy\rho)}{2(1-\rho^2)}\},/\rho/<1.$$

It is known (see, for example [2]) that $f(x,y)=f(x)f(y/x)=f(y)f(x/y),$ where $f(x)$ and $f(y)$ are the standard normal distribution and

$$f(x/y)=\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\{-\frac{(x-\rho y)^2}{2(1-\rho^2)}\}, f(y/x)=\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\{-\frac{(y-\rho x)^2}{2(1-\rho^2)}\}.$$

Let

$$\mathrm{A}_{++}=\{(x,y)\in R^2 : x\geq 0, y\geq 0\}, \mathrm{A}_{--}=\{(x,y)\in R^2 : x<0, y<0\},$$
$$\mathrm{A}_{+-}=\{(x,y)\in R^2 : x\geq 0, y<0\}, \mathrm{A}_{-+}=\{(x,y)\in R^2 : x<0, y\geq 0\},$$

and

$$g(x,y)=C_+^2 I_{\mathrm{A}_{++}}(x,y)f(x,y)+C_-^2 I_{\mathrm{A}_{--}}(x,y)f(x,y)+I_{\mathrm{A}_{+-}}(x,y)u_+(x)f(x)u_-(y)f(y)+ I_{\mathrm{A}_{-+}}(x,y)u_-(x)f(x)u_+(y)f(y), \quad (1)$$

Where $I_{\mathrm{A}}(x,y)$ is the indicator of A and

$$u_+(x)=\alpha^{-\frac{1}{2}}C_+\int_0^\infty f(x/y)dy, \text{ if } x\geq 0 \text{ and } u_+(x)=0, \text{ if } x<0, \quad (2)$$

$$u_-(x)=\alpha^{-\frac{1}{2}}C_-\int_{-\infty}^0 f(x/y)dy, \text{ if } x<0 \text{ and } u_+(x)=0, \text{ if } x\geq 0. \quad (3)$$

Here

$$\alpha=\int_0^\infty\int_0^\infty f(x,y)dxdy=\int_{-\infty}^0\int_{-\infty}^0 f(x,y)dxdy \quad (4)$$

(the values of $\alpha=\alpha(\rho)$ can be found from tables of standard bivariate normal distribution, see, for example, [3]-[5]);

C_+ and C_- are some constants.

Denote

$$A_+=\int_0^\infty u_+(x)f(x)dx, A_-=\int_{-\infty}^0 u_-(x)f(x)dx. \quad (5)$$

After substitution $u_+(x)$ and $u_-(x)$ from (2) and (3) in (5) we obtain

$$A_+=C_+\alpha^{\frac{1}{2}} \text{ and } A_-=C_-\alpha^{\frac{1}{2}}. \quad (6)$$

Let us choose C_+ and C_- in such a way that $C_+>0,\ C_->0$ and

$$C_+ + C_- = \alpha^{-\frac{1}{2}}$$

(for example, $C_+ = C_- = \frac{1}{2\alpha^{\frac{1}{2}}}$ or $C_+ = \frac{1}{3\alpha^{\frac{1}{2}}}$, $C_- = \frac{2}{3\alpha^{\frac{1}{2}}}$). Then from (6):

$$A_+ + A_- = 1 \tag{7}$$

and if

$$g(x) = u(x) f(x), \tag{8}$$

where

$$u(x) = \begin{cases} u_+(x) & if \;\; x \geq 0, \\ u_-(x) & if \;\; x < 0, \end{cases} \tag{9}$$

we have

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty} u(x)f(x)dx = \int_{-\infty}^{0} u(x)f(x)dx + \int_{0}^{\infty} u(x)f(x)dx = \int_{-\infty}^{0} u_-(x)f(x)dx + \int_{0}^{\infty} u_+(x)f(x)dx = A_- + A_+ = 1.$$

Therefore $g(x)$ is a probability distribution density.

Now we can verify that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)dxdy = 1,$$

where $g(x,y)$ is defined by (1).

Really, from (1)-(4)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)dxdy = (C_+^2 + C_-^2)\alpha + 2\alpha C_+C_- = \alpha(C_+^2 + C_-^2 + 2C_+C_-) = \alpha(C_+ + C_-)^2 = 1.$$

We show that marginal distribution densities of $g(x,y)$ are $g(x) = u(x)f(x)$ and $g(y) = u(y)f(y)$:

$$\int_{-\infty}^{\infty} g(x,y)dy = I_{[0,\infty)}(x)\int_{-\infty}^{\infty} g(x,y)dy + I_{(-\infty,0)}(x)\int_{-\infty}^{\infty} g(x,y)dy =$$

$$I_{[0,\infty)}(x)[C_+^2 f(x)\int_{0}^{\infty} f(y/x)dy + u_+(x)f(x)A_-] + I_{(-\infty,0)}(x)[C_-^2 f(x)\int_{-\infty}^{0} f(y/x)dy + u_-(x)f(x)A_+] =$$

$$I_{[0,\infty)}(x)[C_+\alpha^{\frac{1}{2}}u_+(x)f(x) + u_+(x)f(x)A_-] + I_{(-\infty,0)}(x)[C_-\alpha^{\frac{1}{2}}u_-(x)f(x) + u_-(x)f(x)A_+] =$$

$$I_{[0,\infty)}(x)[u_+(x)f(x)(C_+\alpha^{\frac{1}{2}} + A_-)] + I_{(-\infty,0)}(x)[u_-(x)f(x)(C_-\alpha^{\frac{1}{2}} + A_+)] =$$

$$I_{[0,\infty)}(x)u_+(x)f(x) + I_{(-\infty,0)}(x)u_-(x)f(x) = g(x).$$

Similarly

$$\int_{-\infty}^{\infty} g(x,y)dx = g(y).$$

From (1)-(3), (8), (9) it is clear that $g(x,y) = g(x)g(y)$ on $A_{+-} \bigcup A_{-+}$ and really

$$g(x,y) = C_+^2 I_{A_{++}}(x,y)f(x,y) + C_-^2 I_{A_{--}}(x,y)f(x,y) + I_{A_{+-}}(x,y)g(x)g(y) + I_{A_{-+}}(x,y)g(x)g(y),$$

Thus we have proved the folloving

Theorem. *The real function* $g(x,y)$ *defined on* R^2 *by (1) is the probability distribution density with marginal distribution densities* $g(x)$ *and* $g(y)$ *of same form, defined from (8).The random variables X and Y with this joint distribution density* $f_{XY}(x,y) = g(x,y)$ *are* $A_{+-} \bigcup A_{-+} -$ *independent.*

3. Regression.

Suppose the random variables X and Y are $A_{+-} \bigcup A_{-+} -$ independent and have the joint probability distribution density $f_{XY}(x,y) = g(x,y)$, where $g(x,y)$ is defined by (1). It is not difficult to obtain the conditional density

$$f_X(x/Y=y) = \frac{g(x,y)}{g(y)} = C_+^2 I_{A_{++}}(x,y)\frac{f(x,y)}{u_+(y)f(y)} + C_-^2 I_{A_{--}}(x,y)\frac{f(x,y)}{u_-(y)f(y)} + I_{A_{+-}}(x,y)u_+(x)f(x) +$$

$$I_{A_{-+}}(x,y)u_-(x)f(x) = C_+^2 I_{A_{++}}(x,y)\frac{f(x/y)}{u_+(y)} + C_-^2 I_{A_{--}}(x,y)\frac{f(x,y)}{u_-(y)} + I_{A_{+-}}(x,y)u_+(x)f(x) +$$

$$I_{A_{-+}}(x,y)u_-(x)f(x) = I_{A_{++}}(x,y)C_+\alpha^{\frac{1}{2}}\frac{f(x/y)}{\int_0^{\infty} f(u/y)du} + I_{A_{--}}(x,y)C_-\alpha^{\frac{1}{2}}\frac{f(x/y)}{\int_{-\infty}^{0} f(u/y)du} +$$

$$I_{A_{+-}}(x,y)C_+\alpha^{-\frac{1}{2}}f(x)\int_0^{\infty} f(u/x)du + I_{A_{-+}}(x,y)C_-\alpha^{-\frac{1}{2}}f(x)\int_{-\infty}^{0} f(u/x)du.$$

So

$$f_X(x/Y=y) = I_{A_{++}}(x,y)C_+\alpha^{\frac{1}{2}}\frac{f(x/y)}{\int_0^{\infty} f(u/y)du} + I_{A_{--}}(x,y)C_-\alpha^{\frac{1}{2}}\frac{f(x/y)}{\int_{-\infty}^{0} f(u/y)du} +$$

$$I_{A_{+-}}(x,y)C_+\alpha^{-\frac{1}{2}}f(x)\int_0^{\infty} f(u/x)du + I_{A_{-+}}(x,y)C_-\alpha^{-\frac{1}{2}}f(x)\int_{-\infty}^{0} f(u/x)du \tag{10}$$

and

$$f_Y(y/X=x) = I_{A_{++}}(x,y)C_+\alpha^{\frac{1}{2}}\frac{f(y/x)}{\int_0^{\infty} f(u/x)du} + I_{A_{--}}(x,y)C_-\alpha^{\frac{1}{2}}\frac{f(y/x)}{\int_{-\infty}^{0} f(u/x)du} +$$

$$I_{A_{+-}}(x,y)C_+\alpha^{-\frac{1}{2}}f(y)\int_0^{\infty} f(u/y)du + I_{A_{-+}}(x,y)C_-\alpha^{-\frac{1}{2}}f(y)\int_{-\infty}^{0} f(u/y)du. \tag{11}$$

From these expressions we see that on the set $A_{+-}\bigcup A_{-+}$ the conditional distribution density $f_X(x/Y=y)$ is the function only of x and $f_Y(y/X=x)$ is the function only of y. It is natural because x and y are independent on this set .

Using (10) and (11) we find regression of X on Y and Y on X.

Regression of X on Y:

$$E(X/Y=y)=\int_{-\infty}^{\infty} xf_X(x/Y=y)dx = I_{[0,\infty)}(y)C_+\alpha^{\frac{1}{2}}\frac{\int_0^{\infty} xf(x/y)dx}{\int_0^{\infty} f(u/y)du}+$$

$$I_{(-\infty,0)}(y)C_-\alpha^{\frac{1}{2}}\frac{\int_{-\infty}^{0} xf(x/y)dx}{\int_{-\infty}^{0} f(u/y)du}+I_{[0,\infty)}(y)C_+\alpha^{-\frac{1}{2}}\int_0^{\infty} xf(x)(\int_0^{\infty} f(u/x)du)dx+$$

$$I_{(-\infty,0)}(y)C_-\alpha^{-\frac{1}{2}}\int_{-\infty}^{0} xf(x)(\int_{-\infty}^{0} f(u/x)du)dx. \qquad (12)$$

Note that here

$$\int_0^{\infty} f(u/y)du=\frac{1}{\sqrt{2\pi(1-\rho^2)}}\int_0^{\infty} e^{-\frac{(u-\rho y)^2}{2(1-\rho^2)}}du=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\rho y}{\sqrt{1-\rho^2}}}^{\infty} e^{-\frac{u^2}{2}}du=1-\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}}),$$

$$\int_{-\infty}^{0} f(u/y)du=\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}}),$$

$$\int_0^{\infty} uf(u/y)du=-(1-\rho^2)\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2y^2}{2(1-\rho^2)}}+\rho y\left(1-\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}})\right), \qquad (13)$$

$$\int_{-\infty}^{0} uf(u/y)du=(1-\rho^2)\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2y^2}{2(1-\rho^2)}}+\rho y\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}}),$$

Denote

$$K_+=C_+\alpha^{-\frac{1}{2}}\int_0^{\infty} xf(x)(\int_0^{\infty} f(u/x)du)dx,$$

$$K_-=C_-\alpha^{-\frac{1}{2}}\int_{-\infty}^{0} xf(x)(\int_{-\infty}^{0} f(u/x)du)dx \qquad (14)$$

It is clear that

$$K_+ = K_+(\rho) = \int_0^\infty xg(x)dx, K_- = K_-(\rho) = \int_{-\infty}^0 xg(x)dx,$$

and

$$K_+ + K_- = \int_{-\infty}^\infty xg(x)dx = EX.$$

Using (13) and (14) from (12) we obtain:

$$\begin{aligned} E(X/Y=y) = I_{[0,\infty)}(y)C_+\alpha^{\frac{1}{2}}[(\rho^2-1))\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2y^2}{2(1-\rho^2)}} + \\ \rho y\left(1-\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}})\right)][1-\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}})]^{-1} + \\ I_{(-\infty,0)}(y)C_-\alpha^{\frac{1}{2}}[(1-\rho^2))\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2y^2}{2(1-\rho^2)}} + \rho y\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}})][\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}})]^{-1} + \\ I_{(-\infty,0)}(y)K_+ + I_{[0,\infty)}(y)K_-. \end{aligned} \quad (15)$$

Similarly

$$\begin{aligned} E(Y/X=x) = I_{[0,\infty)}(x)C_+\alpha^{\frac{1}{2}}[(\rho^2-1))\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2x^2}{2(1-\rho^2)}} + \rho y\left(1-\Phi(\frac{-\rho x}{\sqrt{1-\rho^2}})\right)][1-\Phi(\frac{-\rho x}{\sqrt{1-\rho^2}})]^{-1} + \\ I_{(-\infty,0)}(x)C_-\alpha^{\frac{1}{2}}[(1-\rho^2))\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2x^2}{2(1-\rho^2)}} + \rho y\Phi(\frac{-\rho x}{\sqrt{1-\rho^2}})][\Phi(\frac{-\rho x}{\sqrt{1-\rho^2}})]^{-1} + I_{(-\infty,0)}K_+ + I_{[0,\infty)}(x)K_-. \end{aligned} \quad (16)$$

Representations (15) and (16) show that regression of X on Y and regression of Y on X are not linear.

Remark. Denote

$$f_+(x/y) = \begin{cases} \dfrac{f(x/y)}{\int_0^\infty f(u/y)du}, & x \geq 0, \\ 0, & x < 0 \end{cases} \quad (17)$$

and

$$f_-(x/y) = \begin{cases} \dfrac{f(x/y)}{\int_{-\infty}^0 f(u/y)du}, & x < 0, \\ 0, & x \geq 0. \end{cases} \quad (18)$$

Then we can rewrite (10) in the following form

$$f_X(x/Y=y)=I_{A_{++}}(x,y)C_+\alpha^{\frac{1}{2}}f_+(x/y)+I_{A_{--}}(x,y)C_-\alpha^{\frac{1}{2}}f_-(x/y)+$$
$$I_{A_{+-}}(x,y)C_+\alpha^{-\frac{1}{2}}f(x)\int_0^{\infty}f(u/x)du+I_{A_{-+}}(x,y)C_-\alpha^{-\frac{1}{2}}f(x)\int_{-\infty}^{0}f(u/x)du. \quad (19)$$

Note that $f_+(x/y)$ and $f_-(x/y)$ defined by (17) and (18) are conditional densities.

3. Joint characteristic function.

Let

$C_+=C_-=\dfrac{1}{2\alpha^{\frac{1}{2}}}$, then from (1):

$$g^*(x,y)=\frac{1}{4\alpha}I_{A_{++}}(x,y)f(x,y)+\frac{1}{4\alpha}I_{A_{--}}(x,y)f(x,y)+I_{A_{+-}}(x,y)g^*(x)g^*(y)+I_{A_{-+}}(x,y)g^*(x)g^*(y), \quad (20)$$

where $g^*(x)$ and $g^*(y)$ are defined by (8), (9), (2) and (3) with $C_+=C_-=\dfrac{1}{2\alpha^{\frac{1}{2}}}$.

Denote

$$\beta=\int_0^{\infty}g^*(x)dx,\gamma=\int_{-\infty}^{0}g^*(x)dx \quad (\beta+\gamma=1). \quad (21)$$

And rewrite (20) in the following form

$$g^*(x,y)=I_{A_{++}}(x,y)\frac{1}{4}\frac{f(x,y)}{\alpha}+I_{A_{--}}(x,y)\frac{1}{4}\frac{f(x,y)}{\alpha}+I_{A_{+-}}(x,y)\beta\gamma\frac{g^*(x)}{\beta}\frac{g^*(y)}{\gamma}+$$
$$I_{A_{-+}}(x,y)\beta\gamma\frac{g^*(x)}{\gamma}\frac{g^*(y)}{\beta}. \quad (22)$$

Note that here

$$\alpha=\int_0^{\infty}\int_0^{\infty}f(x,y)dxdy=\int_{-\infty}^{0}\int_{-\infty}^{0}f(x,y)dxdy.$$

Let the random variables U and V defined on (Ω,F,P) with values in $A_{++}\bigcup A_{--}$ have the joint probability distribution density

$$f_{UV}(x,y)=\frac{f(x,y)}{2\alpha},\quad (x,y)\subset A_{++}\bigcup A_{--},$$
$$f_{UV}(x,y)=0,\qquad (x,y)\subset A_{+-}\bigcup A_{-+}. \quad (23)$$

Assume that the random variables ξ and η with values in $[0,\infty)$ and in $(-\infty,0)$, respectively, have the probability distribution densities :

$$f_\xi(x)=\frac{g^*(x)}{\beta}, x\geq 0,$$
$$f_\xi(x)=0, x<0 \tag{24}$$

and

$$f_\eta(x)=\frac{g^*(x)}{\gamma}, \quad x<0,$$
$$f_\eta(x)=0, \quad x\geq 0. \tag{25}$$

Then, it is clear that the joint characteristic function corresponding to $g^*(x,y)$ has the form

$$\varphi(z_1,z_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(z_1x+z_2y)}g^*(x,y)dxdy=\frac{1}{2}\varphi_{UV}(z_1,z_2)+\beta\gamma[\varphi_\xi(z_1)\varphi_\eta(z_2)+\varphi_\xi(z_2)\varphi_\eta(z_1)], \tag{26}$$

where $\varphi_{UV}(z_1,z_2)$ is the characteristic function corresponding to joint density (23), $\varphi_\xi(z)$ and $\varphi_\eta(z)$ are the characteristic functions corresponding to densities (24) and (25) respectively.

Remark 2. If in (26) $z_1=0$ and $z_2=0$, we obtain $\beta\gamma=\frac{1}{4}$. Really in this case, when $C_+=C_-=\frac{1}{2\alpha^{\frac{1}{2}}}$ we have:

$$\beta=\int_0^\infty g^*(x)dx=\int_0^\infty u^*{}_+(x)f(x)dx=\frac{1}{2\alpha}\int_0^\infty(\int_0^\infty f(y/x))dy)f(x)dx=\frac{1}{2\alpha}\int_0^\infty\int_0^\infty f(x,y)dxdy=\frac{1}{2}$$

and

$$\gamma=\int_{-\infty}^0 g^*(x)dx=\int_{-\infty}^0 u^*{}_-(x)f(x)dx=\frac{1}{2\alpha}\int_{-\infty}^0(\int_{-\infty}^0 f(y/x))dy)f(x)dx=\frac{1}{2\alpha}\int_{-\infty}^0\int_{-\infty}^0 f(x,y)dxdy=\frac{1}{2}.$$

Therefore $\beta=\gamma=\frac{1}{2}$ and $\beta\gamma=\frac{1}{4}$.

მათემატიკა

ნაწილობრივ დამოუკიდებული შემთხვევითი სიდიდეები

ო. ღლონტი

ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

სტატიაში განსაზღვრულია X და Y შემთხვევითი სიდიდეების A-დამოუკიდებლობის ცნება და A-დამოუკიდებელი შემთხვევითი სიდიდეების მაგალითი არის აგებული. განხილულია რეგრესია X-სა Y-ზე და Y-სა X-ზე. ნაპოვნია ასეთი შემთხვევითი სიდიდეების მახასიათებელი ფუნქციის სახე.

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