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BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 6, no. 1, 2012

Mathematics

On the Cramer-Rao Inequality in an Infinite Dimensional Space

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ABSTRACT. The Cramer-Rao inequality is obtained in a Banach space by using the technique of smooth measures. The principle of maximum likelihood is formulated. The examples are considered.

Key words: *Cramer-Rao inequality, smooth measure, maximum likelihood principle.*

The current state of infinite dimensional analysis makes it possible to consider many fundamental problems of statistics in more general terms. The theory of smooth measures [1] provides vast opportunities in this direction. In the present paper, the realization of one of such opportunities is illustrated by an example of generalization of the Cramer-Rao (C-R) inequality and formulation of the maximum likelihood principle in the infinite dimensional case. This approach was actually suggested by E. Gobet in [2] and is based on the Malliavin calculus theory [3, 4]. In [5], the Malliavin calculus was used in the finite-dimensional (more precisely, one-dimensional) case. In this paper, we use the technique of smooth functions which enables us to formulate a general approach to problems for the Cramer-Rao inequality and tackle the questions associated with them. In [6], the theory of smooth measures was applied to the estimation of the logarithmic derivative of a measure.

1. The Logarithmic Derivative of a Measure.

Let $\{\Omega, \Im, P\}$ be a complete probability space. Consider a random element $X = X(\omega;\theta)$ with parameter $\theta \in \Theta$, where $\Theta \subset \Xi$ is a subset of a separable real Banach space Ξ with norm $\|\cdot\|_{\Xi}$. Let $X = X(\omega;\theta)$ have values in a linear space E.

The primary goal of statistics is to estimate an unknown parameter θ. The estimation is based on (iid) observations $X_1, X_2, \ldots, X_n, \ldots$ of the given random variable. We must construct a statistical function

$T = T(X_1, X_2, ..., X_n)$ such that estimates the parameter θ.

Usually, in this situation we obtain the sequence of statistical structures $\{\aleph, \Re, (P(\theta;\cdot), \theta \in \Theta)\}$, where $\aleph = E^n$ $(n = 1, 2, ..., \infty)$ is the linear space generated by the sequence of random variables $X_1, X_2, ..., X_n$, $\Re$ is the σ-algebra generated by observable sets and $\{P(\theta;\cdot), \theta \in \Theta\}$ is the family of probability measures (distributions) generated by the vector $Y = (X_1, X_2, ..., X_n)$ with the help of the relation $P(\theta; A) = P(Y^{-1}(A))$, $A \in \Re$. In classical statistics, the structure $\{\aleph, \Re, (P(\theta;\cdot), \theta \in \Theta)\}$ is the main object of investigation.

On the other hand, there are a wide range of problems in which it is more convenient to operate with the function $X = X(\omega; \theta)$ if we impose on it the condition of smoothness (regularity condition) of the parameter θ. Thus this is a good chance to apply the tool of stochastic calculus of variations.

We in fact use two calculi: one is based on studying the properties of the statistical structure $\{\aleph, \Re, (P(\theta;\cdot), \theta \in \Theta)\}$ where the family of measures $P(\theta,\cdot)$ is assumed to be smooth, and the other employs direct stochastic methods for which the object of investigation is the function $X(\omega, \theta)$.

Thus the family of distributions $\{P(\theta; A), \theta \in \Theta, A \in \Re\}$ is interesting for us in terms of smoothness imposed on the two parameters θ and A.

Let us assume that $\aleph$ is a separable real reflective Banach space. For every fixed $\theta \in \Theta$, $P(\theta,\cdot)$ is a positive measure. If $h \in \aleph$ is some vector, then we denote by $P_h(\theta; A)$ the measure obtained by the shift $P_h(\theta; A) = P(\theta; A + h)$. We say that the measure $P(\theta,\cdot)$ is differentiable along the vector h if there exists a bounded linear functional on $\aleph$ denoted by $d_h P(\theta;\cdot)$ such that for every $A \in \Re$ the following equality is true

$$P_h(\theta; A) - P(\theta; A) = d_h P(\theta; A) h + \alpha(\theta, A; h),$$

where $\alpha(\theta, A; h)$ is a function such that $\alpha(\theta, A; th) = o(t)$, $t \in R$.

In the case where $\aleph$ is a separable real Hilbert space with scalar product $(\cdot, \cdot)_\aleph$ and norm $\|h\|_\aleph$, $h \in \aleph$, we write $P_h(\theta; A) - P(\theta; A) = (d_h P(\theta; A), h)_\aleph + \alpha(\theta, A; h)$ and sometimes (when it does not lead to confusion) under the derivative $d_h P(\theta;\cdot)$ we will understand an element of the Hilbert space. Clearly, the function $d_h P(\theta;\cdot) h$ is a σ-additive (alternating) measure on $\Re$.

The function $\psi_\theta(t) = P(\theta, A + th)$ is nonnegative and everywhere differentiable. If $P(\theta; A) = 0$, $A \in \Re$, then $t = 0$ is the point of a minimum for functions $\psi_\theta(t)$. Therefore $d_h P(\theta; A) = 0$. Thus, by the Radon-Nikodym theorem, there is a measurable function $\beta_\theta(x; h)$ such that $\dfrac{d_h P(\theta; dx)}{P(\theta; dx)} = \beta_\theta(x; h)$. This function is called the logarithmic derivative of the measure $P(\theta;\cdot)$ along a vector $h \in \aleph$. The logarithmic derivative $\beta_\theta(x; h)$ is linear on the second argument. A vector h is called an admissible direction for the measure

$P(\theta;\cdot)$. The set of all admissible directions is called an admissible subspace.

Example 1. Let $H_+ \subset H \subset H_-$ be three Hilbert spaces, the enclosure operator $i: H_+ \to H$ be the Hilbert-Schmidt operator. Such a triple is called the Hilbert-Schmidt structure. Let γ_θ be a Gaussian measure in γ_θ with the correlation operator equal to I in H and an average θ, $\theta \in H_-$. If $h \in H_+$ then the logarithmic derivative of the measure γ_θ along h is $(\theta - x, h)_H$. ■

In the theory of differentiable measures the fact that the integration by parts formula is fulfilled is very important. Let $\aleph$ be a separable real Hilbert space and $f(x)$ be a functional on it. Suppose that $f(x)$ has a derivative along a vector $h \in \aleph : d_h f(x) = \lim_{t\to 0} t^{-1}\left[f(x+th) - f(x)\right]$ and $d_h f(\cdot) \in L_1(P(\theta;\cdot))$ for fixed $\theta \in \Theta$. In that case, if the measure $P(\theta;\cdot)$ is differentiable along h, then

$$\int_{\aleph}\left(d_h f(x), h\right)_{\aleph} P(\theta;dx) = -\int_{\aleph} f(x)\, d_h P(\theta;dx) = -\int_{\aleph} f(x)\beta_\theta(x;h)P(\theta;dx). \tag{1}$$

We can define the logarithmic derivative along some changeable direction (the so-called logarithmic gradient). Equality (1) can be considered on the basis of such a definition or we can act as when defining the derivative measure along a constant direction.

Let $z(x): \aleph \to \aleph$ be a differentiable vector field with a bounded derivative $\sup_{x\in\aleph}\|z'(x)\| < \infty$. Denote the integral stream corresponding to $z(x)$ by S_t, $t \in R$. This means that $\frac{dS_t}{dt} = z(S_t)$, $S_0 = I$.

According to the transformation $P_t(\theta;A) = P\left(\theta; S_t^{-1}(A)\right)$, $A \in \Re$, to the family of measures $(P(\theta;\ \cdot\ ), \theta \in \Theta)$ there corresponds a class of measures $(P_t(\theta;\cdot), \theta \in \Theta, t \in R)$. The measure $P(\theta;\cdot)$ is differentiable along the vector field $z(x)$ if there is a measure (necessarily alternating) $D_z P(\theta; A)$ such that for any bounded and differentiable function $\varphi: \aleph \to R$, $\varphi \in C^1(\aleph; R)$ we have

$$\int_{\aleph}\varphi(x) D_z P(\theta;dx) = -\int_{\aleph}\varphi'(x) z(x) P(\theta;dx).$$

If $D_z P(\theta;\cdot) \ll P(\theta;\cdot)$, then the Radon-Nikodym density is called the logarithmic derivative of $P(\theta;\cdot)$ along the vector field $z(x): \beta_\theta(x;z) = \frac{D_z P(\theta;dx)}{P(\theta;dx)}$.

Let H be embedded in the Hilbert space $\aleph$, where the embedment operator is the Hilbert-Schmidt operator. Then we can consider the Hilbert-Schmidt structure $\aleph^* \subset H \subset \aleph$. Let us choose a class of measures £ for which there exists a measurable, locally bounded function $\lambda: \aleph \to \aleph$ such that for each constant direction $h \in \aleph^*$ there exists a logarithmic derivative along h which has the form $\beta_\theta(x;h) = \lambda(\theta;x)h = (\lambda(\theta,x), h)_H$. In this case we say that the measure has the logarithmic gradient $\lambda(\theta;x)$. If $P(\theta) \in$ £ and the vector field $z: \aleph \to \aleph^*$ and its derivative is bounded, then the measure $P(\theta)$

has a logarithmic gradient and

$$\beta_\theta\left(x;z(x)\right)=\left\langle\lambda(\theta;x),z(x)\right\rangle+trz'(x).$$

By the principle of continuity, this functional can be extended to smooth vector fields $z(x):\aleph\to H$.

Example 2. In the conditions of Example 1, consider the vector field $z(x):H_-\to H_-$ with a bounded derivative $\sup_{x\in H_-}\|z'(x)\|<\infty$. If $z:H_-\to H$, then the logarithmic gradient exists and $\beta_\theta(x;z)=\left(\theta-x,z(x)\right)_H+trz'(x)$.

We need to show the smoothness of measures with respect to the parameter. Assume as above that we have $\left\{\aleph,\Re,\left(P(\theta;\cdot),\theta\in\Theta\right)\right\}$, where $\aleph$ is a separable real Banach space, and Θ is a smooth manifold embedded into another separable real Banach space Ξ. For every fixed $A\in\Re$ and a vector $\vartheta\in\Xi$, consider the derivative of the function $\tau(\theta)=P(\theta;A)$ at a point θ along ϑ. Denote this derivative by $d_\theta P(\theta;A)\vartheta$. For fixed θ and ϑ, it is an alternating measure. It is easy to see that $d_\theta P(\theta,\cdot)\vartheta\ll P(\theta,\cdot)$ and, by the Radon-Nikodym theorem, there exists a measurable function $l_\theta(x;\vartheta)=\dfrac{d_\theta P(\theta;dx)\vartheta}{P(\theta;dx)}$. The function $l_\theta(x;\vartheta)$ is called the logarithmic derivative of the measure $P(\theta,\cdot)$ with respect to the parameter.

When Ξ is a separable Hilbert space, we denote by $\mathcal{K}$ the space of measures, for which the logarithmic derivative with respect to the parameter is represented as the scalar product $l_\theta(x;\vartheta)=\left(\Bbbk(x,\theta),\vartheta\right)_\Xi$. Thus we call $\Bbbk(x,\theta)$ a vector logarithmic gradient. For Examples 1 and 2 we have $\lambda(x,\theta)=\theta-x$ and $\Bbbk(x,\theta)=x-\theta$.

For the family of measures $\left(P(\theta;\cdot),\theta\in\Theta\right)$ with the logarithmic derivative with respect to the parameter along ϑ there exists a measure ν dominating this family. As is known [7], all measures $P(\theta;\cdot)$ are equivalent to one another and $\dfrac{P(\theta_2;dx)}{P(\theta_1;dx)}=\exp\displaystyle\int_{\theta_1}^{\theta_2}l_\theta(x;\vartheta)d\theta$.

2. Regularity Conditions

We present the following regularity conditions.

Condition I. For $X(\theta)=X(\theta;\omega):\Theta\times\Omega\to\aleph$ there is a derivative $X'(\theta)$ on θ along $\vartheta\in\Xi_0$, where $\Xi_0\subset\Xi$ is a subspace of Ξ. It is a linear mapping $\Xi\to\aleph$ for every $\theta\in\Theta$. Thus for any $\vartheta\in\Xi_0$ and $\theta\in\Theta$ we have $\|X'(\theta)\vartheta\|_\aleph\in L_2(\Omega,P)$.

Condition II. $E\left\{X'(\theta)\vartheta\mid X(\theta)=x\right\}$ is a strongly continuous function of *x* for any $\vartheta\in\Xi_0$, $\theta\in\Theta$.

Condition III. The family of measures $\left(P(\theta;\cdot),\theta\in\Theta\right)$ has a logarithmic derivative with respect to the parameter along a constant direction from a subspace $\Xi_0\subset\Xi$ and

$$\beta_\theta(x;h) \in L_2(\aleph, P(\theta)), \quad \vartheta \in \Xi_0, \quad \theta \in \Theta .$$

Condition IV. The family of measures $(P(\theta;\cdot), \theta \in \Theta)$ has a logarithmic derivative with respect to the parameter along a constant direction of subspace $h \in \aleph_0$ and

$$\beta_\theta(x;h) \in L_2(\aleph, P(\theta)), \ h \in \aleph_0, \ \theta \in \Theta .$$

Condition V. For the statistic $T = T(x): \aleph \to R$ the following equality is valid

$$d_\vartheta \int_\aleph T(x) P(\theta;dx) = \int_\aleph T(x) d_\vartheta P(\theta;dx).$$

Lemma. *Under the regularity conditions* I-IV *for the logarithmic derivatives* $\beta_\theta(x;h)$ *and* $l_\theta(x;\vartheta)$ *the following equality is true*

$$l_\theta(x;\vartheta) = -\beta_\theta\left(x; K_{\theta,\vartheta}(x)\right), \tag{2}$$

wher*e*

$$K_{\theta,\vartheta}(x) = E\left\{\frac{d}{d\theta} X(\theta)\vartheta \mid X(\theta) = x\right\}.$$

Proof. By definition, $P(\theta;A) = P\left(X^{-1}(\theta;A)\right)$. Let $f(x)$ be a bounded, continuously differentiable along $h \in \aleph$ real function. Using the formula of the change of variables, we obtain

$$\int_\aleph f(x) P(\theta;dx) = Ef\left(X(\theta)\right).$$

Consider the derivatives on both sides of θ along ϑ. Then we obtain

$$\int_\aleph f(x) d_\theta P(\theta;dx)\vartheta = E\frac{d}{dx} f\left(X(\theta)\right)\frac{d}{d\theta} X(\theta)\vartheta$$

or

$$\int_\aleph f(x) l_\theta(x;\vartheta) P(\theta;dx) = \int_\aleph f'(x) E\left\{X'(\theta)\vartheta \mid X(\theta) = x\right\} P(\theta;dx).$$

Denote $K_{\theta,\vartheta}(x) = E\left\{\frac{d}{d\theta} X(\theta)\vartheta \mid X(\theta) = x\right\}$, then we write

$$\int_\aleph f'(x) K_{\theta,\vartheta}(x) P(\theta;dx) = -\int_\aleph f(x) \beta_\theta\left(x, K_{\theta,\vartheta}(x)\right) P(\theta;dx).$$

Since $f(x)$ is arbitrary, we obtain (2). ■

3. The Cramer-Rao Inequality

Let $\{\aleph, \Re, (P(\theta, \cdot), \theta \in \Theta)\}$ be the statistical structure corresponding to a random element $X(\omega) = X(\theta,\omega)$. Here $\aleph$ is a separable real reflective Banach space, $\Re$ is a σ-algebra of Borel sets, $\Theta \subset \Xi$ is an open subset of the separable real Banach space Ξ. Assume that regularity conditions I-IV are fulfilled.

Suppose that $g(\theta)=E_\theta(T(X))$, where $T:\aleph\to R$ is a measurable mapping (a statistical function). For the statistical function we have one more regularity condition.

Theorem 1 (Cramer-Rao inequality). *Let regularity conditions I-V be fulfilled. Then*

$$Var\,T(X)\ge\frac{(g'_\vartheta(\theta))^2}{E_\theta l_\theta^2(X;\vartheta)}. \tag{3}$$

Proof. Consider the derivative of $g(\theta)$ along $\vartheta\in\Xi$. Then

$$d_\vartheta E_\theta T(X)=d_\vartheta\int_\aleph T(x)P(\theta;dx)=\int_\aleph T(x)d_\vartheta P(\theta;dx)\vartheta=$$

$$=\int_\aleph T(x)l_\theta(x,\vartheta)P(\theta;dx)=E_\theta T(X)l_\theta(X;\vartheta).$$

Thus

$$d_\vartheta E_\theta T(X)=E_\theta T(X)l_\theta(X;\vartheta). \tag{4}$$

Put $T(x)=1$ in (4). We obtain $E_\theta l_\theta(X;\vartheta)=0$. Therefore

$$d_\vartheta E_\theta(T(X))=E_\theta((T(X)-g(\theta))l_\theta(X;\vartheta))$$

and

$$(d_\vartheta E_\theta(T(X)))^2\le E_\theta(T(X)-g(\theta))^2\cdot E_\theta l_\theta^2(X;\vartheta).$$

Hence

$$Var\,T(X)\ge\frac{(g'_\vartheta(\theta))^2}{E_\theta\beta_\theta^2(X;E(X'(\theta)\vartheta\mid X))}.\ \blacksquare$$

Therefore

4. Maximum Likelihood Principle

Let $\{\aleph,\Re,(P(\theta),\theta\in\Theta)\}$ be the statistical structure corresponding to a random element $X=X(\theta)=X(\theta,\omega)$, $\omega\in\Omega$, where $\aleph$ is a separable real Banach space, $\Re$ is a σ-algebra of Borel subsets, Θ is an open subset of another separable real Banach space Ξ. Assume that the family of measures $(P(\theta,\cdot),\theta\in\Theta)$ has a logarithmic derivative $l_\theta(x,\vartheta)$ with respect to the parameter along $\vartheta\in\Xi$. Then, according to Theorem 1, there is a logarithmic derivative with respect to the measure and

$$l_\theta(x,\vartheta)=-\beta_\theta(x,E\{X'(\theta)\vartheta\mid X(\theta)=x\}).$$

Consider the structure of the repeated sample

$$\{\aleph,\Re,(P(\theta),\theta\in\Theta)\}^n=\{\aleph^n,\Re^n,(P^n(\theta),\theta\in\Theta)\}.$$

Theorem 2. If the logarithmic derivative $l_\theta(x,\vartheta)$ with respect to the parameter exists in the statistical structure $\{\aleph,\Re,(P(\theta),\theta\in\Theta)\}$, then there also exists the logarithmic derivative $L_\theta((x_1,...,x_n),\vartheta^n)$ with respect to the parameter along $\vartheta^n \overset{def}{=} (\vartheta,...,\vartheta)$ for the structure $\{\aleph,\Re,(P(\theta),\theta\in\Theta)\}^n$ of the repeated sample and

$$L_\theta((x_1,...,x_n),\vartheta^n)=\sum_{k=1}^{n} l_\theta(x_k,\vartheta)=-\sum_{k=1}^{n}\beta_\theta(x_n,E\{X_k'(\theta)\vartheta \mid X_k(\theta)=x_k\}). \tag{6}$$

Proof. Since there exists $d_\theta^\vartheta P(\theta)$ by condition, it is easy to show that there also exists

$$d_\theta^{\vartheta,...,\vartheta}P^n(\theta)=\sum_{k=1}^{n} d_\theta^\vartheta(x_k,\vartheta)\prod_{\substack{j=1\\ j\neq k}}^{n} P(\theta)$$

which is absolutely continuous with respect to $P^n(\theta)$. The validity of the theorem and formula (6) follow from the Radon-Nikodym theorem. ■

According to Theorem 2, we can formulate the maximum likelihood principle as follows.

Let $X_1,X_2,\ldots,X_n$ be a sample of random variables $X(\theta)$. Here θ is the unknown parameter to be estimated using the sample. Assume that for a distribution $P(\theta)$ of $X(\theta)$, there is a logarithmic derivative $l_\theta(x,\vartheta)$ with respect to the parameter along any vector $\vartheta\in\Xi_0$, and $l_\theta(x,\vartheta)=\langle\lambda(x,\theta),\vartheta\rangle$. Here Ξ_0 is a dense subset of Ξ.

As is known, all measures $P(\theta)$ are equivalent to one another. Let $\theta_0\in\Xi_0$ be a fixed point. Consider the likelihood function $\frac{dP(\theta)}{dP(\theta_0)}(x)=\rho(x,\theta)$.

It is easy to see that if $P\in\mathcal{L}$, then

$$\frac{\rho_\theta'(x,\theta)\vartheta}{\rho(x,\theta)}=l_\theta(x,\vartheta).$$

For the sample $X_1,X_2,\ldots,X_n$ the likelihood function is

$$L(X_1,...,X_n,\theta;\vartheta)=\prod_{k=1}^{n}\rho(X_k,\theta).$$

By the likelihood principle, a value $\theta=\hat{\theta}$ for which the likelihood function takes a maximum value (provided that the parameter θ has such a value) is called a maximum likelihood estimate. Since

$$\ln L(X_1,...,X_n,\theta;\vartheta)=\sum_{k=1}^{n}\ln\rho(X_k,\theta),$$

the condition for a maximum makes it possible to formulate this definition in terms of a logarithmic derivative with respect to the parameter.

A solution (if it exists) of the equation

$$\sum_{k=1}^{n} l_\theta\left(x_k, \vartheta\right) = 0, \qquad \forall \vartheta \in \Xi_0 \tag{8}$$

with respect to θ, is called a maximum likelihood estimate if $\frac{d}{d\theta} l\left(x,\theta\right)$ is negatively determined.

Example 3. Let us consider the sample $X_1, X_2, \ldots, X_n$ of a canonical Gaussian variable with an unknown average θ in the equipped Hilbert space $H_+ \subset H \subset H_-$. Then

$$\beta_\theta(x,h) = \left(\theta - x, h\right)_H, \ h \in H_+.$$

It is obvious that $X\left(\theta\right) = N + \theta$, where N is a canonical Gaussian variable with an average, is equal to 0. $X'\left(\theta\right) = I$, $X'\left(\theta\right)h = h$ and therefore

$$E\left\{X_k'\left(\theta\right)h \mid X_k\left(\theta\right) = x\right\} = h.$$

So (9) becomes

$$\sum_{k=1}^{n}\left(\theta - x_k, h\right)_H = 0.$$

From here

$$\left(\hat{\theta}, h\right)_H = \frac{1}{n}\sum_{k=1}^{n}\left(X_k, h\right)_H, \quad \hat{\theta} = \frac{1}{n}\sum_{k=1}^{n} X_k = \overline{X}$$

and

$$\sum_{k=1}^{n}\frac{d^h}{d\theta}\left(x-\theta, h\right)_H = -n\left\|h\right\|_H^2 \le 0.$$

As an application we consider a random process $x\left(t\right) = \varphi\left(t\right) + w\left(t\right)$ where $w\left(t\right)$ is a standard Wiener process, $\varphi \in C\left[0,\infty\right) = \Xi$ is an unknown component of the observable process. Clearly, $Ex(t) = \varphi(t)$. In this case $H_+ = C'\left[0,\infty\right)$, $H_- = L_2\left[0,\infty\right)$. If $x_1\left(t\right), x_2\left(t\right), \ldots, x_n\left(t\right)$ are observations, then

$$\hat{\varphi}\left(t\right) = \frac{1}{n}\sum_{k=1}^{n} x_k\left(t\right).$$

If $H = R^n$ is of finite dimension, then we obtain a maximum likelihood estimate along any vector $h = \left(h_1, \ldots, h_n\right)$:

$$\hat{\theta} = \left(\hat{\theta}_1, \ldots, \hat{\theta}_m\right) = \left(\frac{1}{n}\sum_{k=1}^{n} X_k^1, \ldots, \frac{1}{n}\sum_{k=1}^{n} X_k^m\right). \ \blacksquare$$

მათემატიკა

კრამერ-რაოს უტოლობის შესახებ უსასრულო განზომილებიან სივრცეში

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გლუვი ზომების ტექნიკის გამოყენებით მიღებულია კრამერ-რაოს უტოლობა ბანახის სივრცეში. ჩამოყალიბებულია მაქსიმალური დასაჯერობის პრინციპი. განხილულია მაგალითები.

REFERENCES

1. *V. V. Bogachev* (2008), Differentsial'nye mery i ischisleniya Mallyavina. Moscow-Izhevsk (in Russian).
2. *E. Gobet* (2001), Bernoulli, **7**(6): 899-912.
3. *P. Malliavin* (1976), In: *Ito K.* (ed.), Proc. of Int. Symp. Stoch. D. Eqs. Kyoto, 195-263.
4. *D. Nualart* (2006), The Malliavin Calculus and Related Topics. Berlin: Springer Verlag, 2nd ed.
5. *J. M. Corcuera, A.Kohatsu-Higa* (2008), Statistical Inference and Malliavin Calculus. Math. Preprint Series, No. 410, IMUB, Barcelona, Spain.
6. *E. Nadaraya, G. Sokhadze* (2010), Georgian Math. J., **17**: 741-747.
7. *Yu. L. Daletskii, G. Sokhadze* (1988), Functional Analysis and Its Applications, **22**, 2: 149-150.

Received December, 2011