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Mathematics

On Reconstruction of Coefficients of a Multiple Trigonometric Series with Lebesgue Nonintegrable Sum

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ABSTRACT. Cantor's functionals sequence notion for one-dimensional trigonometric series is introduced. Also, the possibility of reconstruction of coefficients of multiple trigonometric series with Lebesgue nonintegrable sum by iterated use of Cantor's functionals is established. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *uniqueness of trigonometric series, Cantor's and Valle-Poussin's theorems, Denjoy integral, T-integrals, Cantor's functionals sequence, multiple trigonometric series.*

We denote the trigonometric system defined on [0,1] by $T^1 = \{t_i(\tau)\}_{i=0}^{\infty}$, where $t_0(\tau) \equiv 1$, $t_{2i-1}(\tau) = \sqrt{2}\cos 2\pi i\tau$ and $t_{2i}(\tau) = \sqrt{2}\sin 2\pi i\tau$, $i = 1,2,\ldots$.

Let consider a trigonometric series

$$\sum_{i=0}^{\infty} a_i t_i(\tau). \tag{1}$$

Partial sums of the series (1) will be denoted by

$$S_m(\tau) = \sum_{i=0}^{2m} a_i t_i(\tau).$$

Definition 1. A set $A \subset [0,1]$ belongs to the class $U(T^1)$ if the only series (1) that converges to zero on A is the series all of whose coeffitients are zero.

Definition 2. We say that a function $f(\tau)$ belongs to the class $J(A, T^1)$ if $A \in U(T^1)$ and there exists a series (1) such that equality

$$\sum_{i=0}^{\infty} a_i t_i(\tau) = f(\tau) \qquad (2)$$

holds true for any $\tau \in A$.

Note that if $f \in J(A, T^1)$, then the uniqueness of coefficients of series (2) follows from the definition of $U(T^1)$.

Definition 3. We say that a sequence of functionals $\{G_i^A(f(\tau))\}_{i=0}^{\infty}$ is Cantor's functionals sequence if for any function $f \in J(A, T^1)$ and for every $i = 0,1,2,\ldots$ the equality

$$a_i = G_i^A(f(\tau))$$

holds.

The following theorem, proved by Cantor [1] in 1872 is a fundamental result in the uniqueness theory for trigonometric series.

Theorem A. *If the series (1) is everywhere convergent to zero, then all coefficients of the series are zero.*

This theorem was generalized in various direction. In particular, in 1912 Valle-Poussin [3] proved the following statement.

Theorem B. *If the series* (1) *converges to a finite integrable in the Lebesgue sens, function f everywhere possibly exept at countably many points, then it is the Fourier-Lebesgue series of f.*

The author [2] proved some results concerning the uniqueness of certain multiple function series for Pringsheim convergence. In particular, these results imply the Theorems A and B remain valid for multiple trigonometric series.

It is well-known that there exsits such trigonometric series, that their sums are not integrable functions in the Lebesgue sens. An example of such series is given by the series

$$\sum_{n=2}^{\infty} \frac{\sin 2\pi n\tau}{\ln n}.$$

In the other hand, the uniqueness of coefficients of everywhere convergent trigonometric series follows from Cantor's theorem. This circumstance caused the necessity of such generalization of Lebesgue integral notion, that any everywhere convergent trigonometric series would be Fourier series in the generalized integral sens. This problem was solved by Denjoy. It is known other generalizations of Lebesgue integral notion, which also solve the problem. Examples of such generalization are: MZ Marcinkiewicz-Zygmund integral, P^2 James integral, SCP Burkill integral and other, so called *T*-integrals (see [4]). So the Fourier formulas for coefficients of everywhere convergent trigonometric series

$$a_i = \int_0^1 f(\tau) t_i(\tau) d\tau, \qquad i = 0,1,2,\ldots$$

in every above mentioned generalized integral sens give examples of Cantor's functionals sequences when *A*=[0,1].

Let $d \geq 2$ be an integer, R^d Euclidean space of dimension *d*, Z_0^d the set of all points with nonnegative integer coordinates in R^d. We denote points of the set Z_0^d by $m = (m_1, \ldots, m_d)$ and $n = (n_1, \ldots, n_d)$. We use

$x = (x_1, \ldots, x_d)$ to denote points of the unit cube $[0,1]^d$.

We consider d-multiple trigonometric series

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} a_{n_1,\ldots,n_d} \prod_{j=1}^{d} t_{n_j}(x_j). \tag{3}$$

Rectangular partial sums of the series (3) we denote by $S_m(x)$. i. e.

$$S_m(x) = \sum_{n_1=0}^{2m_1} \cdots \sum_{n_d=0}^{2m_d} a_{n_1,\ldots,n_d} \prod_{j=1}^{d} t_{n_j}(x_j).$$

The convergence of the series(3) will be understood as Pringsheim convergence.

The Cartesian product for every $2 \le p \le d$ and any set A will be denoted by

$$\underbrace{A \times \cdots \times A}_{p} = A^p.$$

Let consider a function $F(x_1, \ldots, x_p)$, where $(x_1, \ldots, x_p) \in A^p$.

We introduce a sumbol $G_i^{A,p}(F(x_1, \ldots, x_p))$ which generalize G_i^A for multivariable situations. Namely, for every $2 \le p \le d$, the symbol $G_i^{A,p}(F(x_1, \ldots, x_{p-1}, x_p))$ means that G_i^A acts on a function $F(x_1, \ldots, x_{p-1}, x_p)$, where only $x_p \in A$ is an independent variable and $(x_1, \ldots, x_{p-1})$ is a fixed point of the set A^{p-1}. Also, $F(x_1, \ldots, x_{p-1}, x_p) \in J(A, T^1)$ for any $(x_1, \ldots, x_{p-1}) \in A^{p-1}$. If $p = 1$, then $G_i^{A,1}(F(x_1)) = G_i^A(F(x_1))$.

We established that it is possible to calculate coefficients of convergent series (3) by iterated using of Cantor's functionals. Namely, the following holds true.

Theorem. *Let* $A \in U(T^1)$ *and* $\{G_i^A(f(\tau))\}_{i=0}^{\infty}$ *is Cantor's functionals sequence and for any* $(x_1, \ldots, x_d) \in A^d$ *the series* (3) *converges to the function* $F(x_1, \ldots, x_d)$*, then for every* $n = (n_1, \ldots, n_d) \in Z_0^d$ *the equality*

$$a_{n_1,\ldots,n_d} = G_{n_1}^{A,1}\left(G_{n_2}^{A,2}\left(\cdots\left(G_{n_d}^{A,d}\left(F(x_1, \ldots, x_d)\right)\right)\cdots\right)\right)$$

holds.

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