On Some Approximation Properties of a Generalized Fejér Integral

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ABSTRACT. Problems of approximation in spaces of $p$-integrable for some $p \geq 1$, as well as essentially bounded functions defined on a locally compact Abelian group are considered. Analogs of Fejér well-known positive operators are taken as approximate aggregates.

Key words: locally compact Abelian group, positive operator, Fejér integral, approximation.

In papers [1-5] problems of approximative nature are considered for some spaces of real or complex valued functions and also measures defined on a locally compact Abelian group. Let $G$ be a locally compact Abelian Hausdorf group and $\hat{G}$ be the dual group, i.e. the set of all characters on $G$. $\hat{G}$ is also a locally compact Abelian group in the topology of uniform convergence of characters on compact subsets of $G$. $U_{\hat{G}}$ will stand for the collection of all symmetric compact sets from $\hat{G}$ which are closures of neighborhoods of the unity in $\hat{G}$. $KT = \{g : g = g_1g_2, \ g_1 \in K, \ g_2 \in T\}$ will stand for the product of the sets $K$ and $T$, while $(1)_K$ will denote the characteristic function of the set $K$. By $L^p(G) = L^p(G, \mu)$, $1 \leq p < \infty$, is denoted the space of $p$-th power integrable functions on $G$ with respect to the Haar measure $\mu$. $L^\infty(G) = L^\infty(G, \mu)$ denotes the space of essentially bounded on $G$ functions with respect to $\mu$. For arbitrary $K$ and $T$ in $\hat{G}$ we consider the following functions defined on $G$ ([6], Ch.5, §1)

$$V_{K,T}(g) = (\text{mes}T)^{-1}(\hat{1})_T(\hat{g})(\hat{1})_{KT}(g).$$ (1)

Here and in the sequel by $\hat{f}$ (resp. $\tilde{f}$) is denoted the Fourier (resp. the inverse) transform of $L^p(G, \mu)$, $1 \leq p \leq 2$. Usually, the Haar measures on $G$ and $\hat{G}$ are normalized so that the inversion formula $f = (\hat{f})^{-1}$ holds for functions $f \in L^1(G)$, $\hat{f} \in L^1(\hat{G})$. 

In connection with the density problem we have introduced a definition of $W^p(K)$, $K \in U_G$ of continuous functions $f \in L^p(G)$, such that

$$f(g) = (f * V_{k,T})(g) = \int_G f(h) V_{k,T}(h^{-1}g) d\mu(h) = \int_G f(h) V_{k,T}(h^{-1}g) dh,$$

for all $g \in G$, $T \in \hat{G}$.

If $G = \mathbb{R}^m$, $m \geq 1$, and $K$ is a symmetric body of $\mathbb{R}^m$, the class $W^p(K)$, $1 \leq p \leq \infty$, coincides with the well-known class of entire functions of exponential type $K$ whose traces on $\mathbb{R}^m$ belong to the space $L^p(\mathbb{R}^m)$ [7, 8]. For the case of compact $G$ the functions from $W^p(K)$, $1 \leq p \leq \infty$, are finite linear combinations of characters of $G$.

In papers [1, 2] it is proved that $W^p(K)$, $1 \leq p \leq \infty$, is the shift invariant closed subspaces of $L^p(G)$. In the case when $1 \leq p \leq 2$, the set $W^p(K)$ coincides with the set $F^p(K)$, which consists of continuous functions on $G$, whose Fourier transform supports belong to $K$. Moreover, the set of functions $W^p(K)$ for all possible compact $K \in U_G$ is dense in $L^p(G)$, $1 \leq p < \infty$. In connection with the density problem we remark the following.

Let $K$ be a symmetric compact set from $\hat{G}$. In [4] we have introduced a definition of $B$-property, which is an analogy of the notion of convexity for locally compact Abelian groups. According to this definition, a set $K$ possesses the $B$-property if the element $g \in \hat{G}$, which admits for a certain natural number $n$ the representation $g^n = g_1^n \cdots g_k^n$, where $n_1, \ldots, n_k$ are natural numbers, $n = n_1 + \cdots + n_k$, while $g_1, \ldots, g_k \in K$, belongs to $K$. If $T$ is a set from $U_G$, without the $B$-property, it can be put in a $K \in U_G$ which does have the $B$-property.

To construct such a minimal set $K$, we must consider all elements $g \in \hat{G}$ which can be represented in a form $g^n = g_1^n \cdots g_k^n$, where $g_1, \ldots, g_k \in T$, $n_1, \ldots, n_k \in N$, $n = n_1 + \cdots + n_k$ and connect them to $T$ such $g$, if $g \not\in T$. If we connect to $T$ all such $g$, then the obtained set $K$ possesses the required property. Really, let us consider elements $g_1, \ldots, g_k \in K$ and suppose that $g^n = g_1^n \cdots g_k^n$, for some $g \in G$, $g_1, \ldots, g_k \in K$, $n_1, \ldots, n_k \in N$, $n = n_1 + \cdots + n_k$. It suffices to consider the case $k = 2$. Let $g_1^n = g_{11}^n \cdots g_{1h}^n$, $g_2^n = g_{21}^n \cdots g_{2s}^n$, $n_1 = n_{11} + \cdots + n_{1h}$, $n_2 = n_{21} + \cdots + n_{2h}$, $g_{1i} \in T$, $i = 1, \ldots, h$, $g_{2j} \in T$, $j = 1, \ldots, l$. Let $n$ be the least common multiple of the numbers $n_1$ and $n_2$, and $n = n_1 m_1 = n_2 m_2$. Let $g = g_1^s g_2^s$, $s = s_1 + s_2$ ($s_1, s_2 \in N$). Let us prove that $g \in K$. We have $g^n = (g^s)^n = g_1^{mn_1} g_2^{mn_2} = g_{11}^{mn_1} \cdots g_{1h}^{mn_1} g_{21} g_{2s}^{mn_2} \cdots g_{2s}^{mn_2}$. Here $n_1 m_1 s_1 + \cdots + n_{1h} m_{1h} s_1 + n_{21} m_{21} s_2 + \cdots + n_{2h} m_{2h} s_2 = ns$, $g_{11}, \ldots, g_{2s} \in T$ and by construction of $K$, we have that $g \in K$. Thus $K$ has the $B$-property. Inclusion $K \in U_G$ is clear. Thus every $T \in U_G$ may be put in some $K \in U_G$, which has the $B$-property. It is shown in [1], that $W^p(T) \subset W^p(K)$, $1 \leq p \leq \infty$, for every $T \subset K$, and therefore we can take the sets $W^p(K)$ as a dense set in $L^p(G)$ for all $K \in U_G$, having the $B$-property too.
Let $I$ be an ordered unbounded set $I \subset \mathbb{R}^+$ and consider a generalized sequence of sets $K \in U_{\hat{G}}$, such that $K_\alpha \subset K_\beta$ if $\alpha < \beta$ ($\alpha, \beta \in I$) and $\bigcup_{\alpha \in I} K_\alpha = \hat{G}$. For a such sequence $L^p(\mathbb{R}^m)$, $1 \leq p \leq \infty$ we have studied in [5] the following sequence of positive operators

$$
\sigma_{K_\alpha}(f)(g) = (f * V_{K_\alpha})(g) = \int_G f(h) V_{K_\alpha}(h^{-1}g) dh,
$$

(2)

where

$$
V_{K_\alpha}(g) = (\operatorname{mes} K_\alpha)^{-1}((1)_{K_\alpha}(g))^2, \quad K_\alpha \in U_{\hat{G}}.
$$

(3)

The kernels (3) are mentioned in [6], (Ch.5, §1) and they represent the limiting case of the kernel $V_{K,T}$ defined by (1), when $K$ converges to the unity of $\hat{G}$ and $T$ is replaced by $K_\alpha$. In [5] it is proved that if $f \in L^p(G)$, $1 \leq p \leq \infty$ then $\sigma_{K_\alpha}(f) \in W^p(K_\alpha^2)$.

Some results concerning positive linear operators and the approximation of continuous functions on locally compact Abelian groups are given in [9]. First of all the positive operators (2) are of importance as in the case $G = \mathbb{R}^m$ they coincide with the well-known Fejér operator. For example, if $m = 1$, $I = (0, \infty)$, $\alpha \in I$ and $K_\alpha = [-\alpha, \alpha]$ , we have the sequence of operators (see, for example, [10], 3.1.2, p. 122)

$$
\sigma_{[-\alpha, \alpha]}(f)(x) = \sigma_\alpha(f)(x) = \frac{2}{\pi \tau} \int_{-\infty}^{\infty} f(t)(x-t)^2 \sin^{\frac{\tau}{2}}(x-t) dt, \quad \tau \in \mathbb{R}^+.
$$

In the case when $G = \mathbb{R}^m$, $m \geq 1$ and $K$ is the ball of a radius $\tau \in \mathbb{R}^+$, the kernel of the operator (2) takes the form

$$
V_k(x) = (\alpha_m)^{-1} |x|^{-m} J_{m/2}^2(\tau |x| / 2), \quad x \in \mathbb{R}^m,
$$

where $J_{m/2}$ is the Bessel function of order $m/2$ and $\alpha_m$ is the area of the $m$-dimensional unit sphere. In that case, the operator (2) is studied in [11] in connection with the saturation problem in $L^p(\mathbb{R}^m)$, $1 \leq p \leq \infty$. In [5] it is proved that if $f \in L^p(G)$, $1 \leq p < \infty$, and the sequence of sets $K_\alpha \in U_{\hat{G}}$ satisfies the condition

$$
\lim_{\alpha \to \infty} \frac{\operatorname{mes}(TK_\alpha)}{\operatorname{mes}(K_\alpha)} = 1
$$

(4)

for all fixed $T \in U_{\hat{G}}$, then

$$
\lim_{\alpha \to \infty} \| f - \sigma_{K_\alpha}(f) \|_{L^p(G)} = 0.
$$

Under the condition (4) the sequence (2) converges to $f$ also in the space $L^\infty(G)$ , but in the weak* topology of $L^\infty(G)$ . An analogous result is valid also in the space $M(G)$ of bounded regular complex valued Borel measures on $G$ [5]. In addition to these results we state the following

**Theorem 1.** Let $\{K_\alpha\}$ be a sequence in $U_{\hat{G}}$ satisfying (4) and $S \subset G$ is a compact set. If a function $f \in L^\infty(G)$ is continuous at a neighborhood of $S$, then $\sigma_{K_\alpha}(f)$ converges to $f$ uniformly on $S$ as $\alpha \to \infty$.

**Proof.** Since $f$ is uniformly continuous on $S$, given $\delta > 0$ find a neighborhood $V \in U_{\hat{G}}$ of the unity of
such that
\[ |f(hg) - f(g)| < \delta / 2 \quad \text{for all } g \in T, \ h \in V. \] (5)

In [5] (Proposition 4) it is proved that under assumption (4) the sequence of kernels \( \{V_{K_\alpha}\} \) represents an approximate unity. This means that for every fixed \( U \in U_{\hat{G}} \) the following equalities hold
\[
\lim_{\alpha \to \infty} \int_{G/V} V_{K_\alpha}(g)dg = 0, \quad \int_G V_{K_\alpha}(g)dg = 1.
\]
Therefore there exists a \( \alpha_0 > 0 \) such that
\[
\int_{G/V} V_{K_\alpha}(g)dg < \frac{\delta}{4\|f\|_{L^p(G)}}, \quad \text{for all } \alpha > \alpha_0. \] (6)

By use of the equality \( \int_G V_{K_\alpha}(g)dg = 1 \) we have

Applying (5) and (6), and representing the integral of the right-hand as \( \int_V + \int_{G/V} \), we obtain
\[
\sup_{g \in T} |f(g) - \sigma_{K_\alpha}(f)(g)| \leq \delta / 2 + \delta / 2 = \delta, \quad \text{when } \alpha \to \infty.
\]

Now we prove that the rate of convergence of the operators \( \sigma_{K_\alpha} \) to the identical operator in general is not greater than \( \text{mes} K_\alpha \).

**Theorem 2.** Let \( f \in L^p(G), \ 1 \leq p \leq 2, \) and a sequence of sets \( K_\alpha \subset U_{\hat{G}} \) satisfy the condition
\[
\limsup_{\alpha \to \infty} \{\text{mes} K_\alpha \backslash \text{mes} (K_\alpha \cap (\chi K_\alpha))\} \neq 0 \] (7)
for any fixed \( \chi \in \hat{G} \). Then from the condition
\[
\|f - \sigma_{K_\alpha}(f)\|_{L^p(G)} = o(\text{mes} K_\alpha), \quad \alpha \to \infty, \] (8)
it follows that \( f(g) = 0 \) a.e. on \( G \).

**Proof.** Since \( f \in L^p(G), \ 1 \leq p \leq 2, \) and \( V_{K_\alpha} \in L^1(G) \) for all \( \alpha \in I \), we have for the Fourier transform of (2) ([12], Ch.8)
\[
(\sigma_{K_\alpha}(f)) \hat{\chi} = \hat{f}(\chi)(V_{K_\alpha}) \hat{\chi}, \] (9)
and
\[
(V_{K_\alpha}) \hat{\chi} = (\text{mes} K_\alpha)^{-1}((1)_{K_\alpha} * (1)_{K_\alpha}) \hat{\chi}.
\]

It is clear that \( (1)_{K_\alpha} * (1)_{K_\alpha} \in L^1(\hat{G}) \) and \( (1)_{K_\alpha} * (1)_{K_\alpha} \hat{\chi} \in L^1(G) \). Therefore we have from (9)
\[
(\sigma_{K_\alpha}(f)) \hat{\chi} = \hat{f}(\chi)(\text{mes} K_\alpha)^{-1}((1)_{K_\alpha} * (1)_{K_\alpha})(\hat{\chi}^{-1})
\]
It follows from here
\[ (\text{mes } K_\alpha)(f - \sigma_{K_\alpha}(f))^\vee (\chi) = \hat{f}(\chi)(\text{mes } K_\alpha - \int_{K_\alpha}(h^{-1}\chi^{-1})dh). \]  (10)

The upper bound of the \( L^p \) norm \((p^{-1} + q^{-1} = 1)\) on the left-hand is, according to the Hausdorf inequality ([12], §31.22), \((\text{mes } K_\alpha) \| f - \sigma_{K_\alpha} \|_{L^q(G)}^p\). Then it follows from (8) and (10)
\[
\lim_{\alpha \to \infty} \| \hat{f}(\chi)(\text{mes } K_\alpha - \int_{K_\alpha}(h^{-1}\chi^{-1})dh) \|_{L^q(G)}^p = 0.
\]

According to (3), this means that \( \| \hat{f} \|_{L^q(G)} = 0\), i.e. \( f(g) = 0 \) a.e. on \( G ([12], \S 31.31) \).

Now we give examples of operators \( \sigma_{K_\alpha} \) for some groups \( G \) and sets \( K_\alpha \):

1. Let \( G = \mathbb{R}^m, m \geq 1 \) and consider the sets \( K_\alpha = \{ u : u \in \mathbb{R}^m, d(u) \leq \alpha \}, \alpha > 0 \), where \( d^2(u) = \langle u, Au \rangle = \sum_{k,l=1}^m a_{k,l}u_ku_l \) and \( A = (a_{k,l})_{k,l=1}^m \) is a positive definite matrix. It is clear that \( K_\alpha \subseteq U^\alpha_G = U_{\mathbb{R}^m} \). It is possible to calculate \( \langle 1 \rangle_{K_\alpha} \) by analogy to [13] (we must keep in mind that according to our agreement the Haar measure \( \mu \) in \( \mathbb{R}^m \) is normalized in such a way that \( \mu(E) = (2\pi)^{m/2}l(E), \quad E \subseteq \mathbb{R}^m \), where \( l(E) \) is the Lebesgue measure of \( E \)). Actually, we can obtain the following representation of \( \sigma_{K_\alpha} \)
\[
\sigma_{K_\alpha}(F)(x) = \frac{2^m\pi^{m/2}\Gamma(m/2 + 1)}{\sqrt{\det A}} \int_{\mathbb{R}^m} f(t + x) J_{m/2}((\alpha/2)\sqrt{(x,A^{-1}x)}) dx,
\]
where \( J_{m/2} \) is the Bessel function of order \( m/2 \) and \( \Gamma \) is the Euler gamma function.

Let us consider the matrix \( B \), whose columns are orthonormal eigenvectors of \( A \). Let \( B' \) be the matrix conjugate to \( B \) and \( A \) be the diagonal matrix composed with the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_m \) of \( A \). It is well known that \( A = B'B, \det A = \det A, \det B = 1 \). The change of variables \( u = Bv, v_k = \lambda_k^{1/2}z_k \) gives
\[
\langle u, Au \rangle = (Bv, BAv) = (v, B'ABv) = (v, Av) = \| z \|^2 = z_1^2 + z_2^2 + \cdots + z_m^2.
\]

Thus \( d(u) \leq \alpha \Leftrightarrow \| z \| \leq \alpha \). This linear correspondence between \( K_\alpha \) and \( C_\alpha := \{ z \in \mathbb{R}^m : \| z \| \leq \alpha \} \) is \( z = (BA)^{-1}u \). By this map the image of \( C_\alpha + \mu \) for any \( \mu \in K_\alpha \) is translation of \( C_\alpha \) by the vector \( \mu' = (BA)^{-1}\mu \). It is clear that \( \text{mes } K_\alpha \setminus \text{mes } (K_\alpha + \mu) = \text{mes } C_\alpha \setminus \text{mes } (C_\alpha + \mu') \). It may be proved by calculation that \( \text{mes } C_\alpha \setminus \text{mes } (C_\alpha + \mu) \geq C(\mu')\alpha^{m-1} \), where \( C(\mu') \) depends only on \( \mu' \) a positive number. Thus, in the considered example condition (7) is satisfied. Therefore Theorem 2 implies that in this example the approximation by operators (11) of order \( \alpha \) of nontrivial functions from \( L^p(\mathbb{R}^m) \), \( 1 \leq p \leq 2 \), is impossible.

2. If \( G \equiv \mathbb{Z} \) is the group of whole numbers, then \( \hat{G} = E \) is the unit circumference from \( \mathbb{R}^2 \) up to an isomorphism. Characters of \( G \) have the form \( \chi_L(n) = e^{it}, n \in \mathbb{Z}, t \in E \). Let \( K_\alpha = \{ e^{i\theta} : -\pi + \pi/\alpha \leq \theta \leq \pi + \pi/\alpha \quad 1 < \alpha < \infty \} \). Take, as a dual measure on \( E \), the arc length divided into \( i\sqrt{2\pi} \). We have
(1)_{K_{\alpha}}(n) = \frac{1}{i\sqrt{2\pi}} \int_{K_{\alpha}} \xi^n d\xi = \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi}} \frac{(\alpha - 1)(n + 1)}{\alpha}, & n \neq -1, \\ \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\alpha - 1}{\alpha}, & n = -1, \end{cases}

and therefore, in this case
\[ \sigma_{K_{\alpha}}(f)(n) = \frac{\alpha}{(\alpha - 1)^2 \pi} \sum_{\xi = k + \alpha, \xi \neq \alpha}^{\infty} f(k) \sin^2 \left( \frac{\alpha - 1}{\alpha} \frac{(n - k + 1)(n + 1)}{(n + 1)^2} \right) + \frac{\alpha}{\pi} f(n + 1). \]

3. Let \( G = \mathbb{R}^+ \) be the multiplicative group of positive integers with the unit \( e = 1 \). This group has a character \( \chi_\xi = \xi^\alpha \), where \( x \in \mathbb{R} \) and \( G = \mathbb{R}^+ \) up to an isomorphism. Take as \( K_{\alpha} \) the interval \( K_{\alpha} = [-\alpha, \alpha] \), \( \alpha \in \mathbb{R}^+ \). Then
\[ (1)_{K_{\alpha}}(\xi) = \int_{-\alpha}^{\alpha} \xi \xi \sin \left( \frac{\alpha \ln \xi}{\ln \xi} \right) d\xi = 2 \sin \left( \frac{\alpha \ln \xi}{\ln \xi} \right) \]

And, therefore, in this case
\[ \sigma_{K_{\alpha}}(f)(x) = \frac{1}{\alpha \pi} \int_{0}^{\infty} f(h) \sin^2 \left( \frac{\alpha \ln \frac{x}{h}}{\ln \frac{x}{h}} \right) dh. \]

4. Let \( G = \mathbb{Q}_p \) be the field of the \( p \)-adic numbers with a prime \( p \). With respect to the additional operation of \( p \)-adic numbers, \( \mathbb{Q}_p \) is a locally compact Abelian group [14]. Its dual group is isomorphic to the addition group \( \mathbb{Q}_p \). The character \( \chi_\xi \), corresponding to a \( p \)-adic number \( \xi \), has the form \( \chi_\xi(x) = \exp(2\pi i \{ \xi x \}^p) \).

Here \( \{ x \}_p \) is defined by the \( p \)-adic expansion \( x = \sum_{n} a_n p^n \) as
\[ \{ x \}_p = \begin{cases} \sum_{n = \text{ord}_p(x)}^{-1} a_n p^n, & \text{if } \text{ord}_p(x) < 0, \\ 0, & \text{if } \text{ord}_p(x) \geq 0, \end{cases} \]

and \( \xi x \) is the product of \( p \)-adic numbers \( \xi \) and \( x \) in the field \( \mathbb{Q}_p \).

Let \( n \in \mathbb{Z} \), \( \alpha = p^n \) and \( K_{\alpha} \) be the \( p \)-adic ball of the radius \( p^n \) with the center in zero, i.e. \( K_{\alpha} = \{ h \in \mathbb{Q}_p : |h|_p \leq p^n \} \), where \( |h|_p \) is the \( p \)-adic norm of \( h \). The Haar measure of this ball is \( \text{mes} K_{\alpha} = p^n \) ([14], §4, (2.3)). By means of the formula ([14], §4, (3.1))
\[ \int_{K_{\alpha}} \chi_\xi(x) dx = \begin{cases} p^n, & \text{if } |\xi|_p < p^{-n}, \\ 0, & \text{if } |\xi|_p > p^{-n+1}, \end{cases} \]

we obtain that in the considered case the operator \( \sigma_{K_{\alpha}}(x) = p^{-n} \int_{\mathbb{Q}_p} f(\xi + x) d\xi \left( \int_{|\xi|_p < p^n} e^{2\pi i \chi_\xi(y)} dy \right)^2 = p^n \int_{|\xi|_p < p^{-n}} f(\xi + x) d\xi \).
მათემატიკა

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REFERENCES


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