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Mathematics

On Some Approximation Properties of a Generalized Fejér Integral

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ABSTRACT. Problems of approximation in spaces of *p*-integrable for some $p \geq 1$, as well as essentially bounded functions defined on a locally compact Abelian group are considered. Analogs of Fejér well-known positive operators are taken as approximate aggregates . *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *locally compact Abelian group, positive operator, Fejér integral, approximation.*

In papers [1-5] problems of approximative nature are considered for some spaces of real or complex valued functions and also measures defined on a locally compact Abelian group. Let G be a locally compact Abelian Hausdorf group and $\hat{G}$ be the dual group, i.e. the set of all characters on G. $\hat{G}$ is also a locally compact Abelian group in the topology of uniform convergence of characters on compact subsets of G. $U_{\hat{G}}$ will stand for the collection of all symmetric compact sets from $\hat{G}$ which are closures of neighborhoods of the unity in $\hat{G}$. $KT = \{g : g = g_1 g_2, \quad g_1 \in K, \quad g_2 \in T\}$ will stand for the product of the sets K and T, while $(1)_K$ will denote the characteristic function of the set K. By $L^p(G) \equiv L^p(G, \mu), \quad 1 \leq p < \infty$, is denoted the space of p-th power integrable functions on G with respect to the Haar measure μ. $L^\infty(G) \equiv L^\infty(G, \mu)$ denotes the space of essentially bounded on G functions with respect to μ. For arbitrary K and T in $\hat{G}$ we consider the following functions defined on G ([6], Ch.5, §1)

$$V_{K,T}(g) = (\text{mes}T)^{-1} (\hat{1})_T(g) (\hat{1})_{KT}(g). \tag{1}$$

Here and in the sequel by $\hat{f}$ (resp. $\tilde{f}$) is denoted the Fourier (resp. the inverse) transform of $L^p(G, \mu), \quad 1 \leq p \leq 2.$ Usually, the Haar measures on G and $\hat{G}$ are normalized so that the inversion formula $f = (\hat{f})^\sim$ holds for functions $f \in L^1(G), \quad \hat{f} \in L^1(\hat{G}).$

In [1] is introduced the set $W^p(K),\quad K\in U_{\hat{G}}$ of continuous functions $f\in L^p(G)$, such that

$$f(g)=(f*V_{K,T})(g)\equiv\int_G f(h)\ \ V_{K,T}(h^{-1}g)d\mu(h)\equiv\int_G f(h)\ \ V_{K,T}(h^{-1}g)dh,$$

for all $g\in G,\ \ T\in\hat{G}.$

If $G=\mathbb{R}^m,\quad m\geq 1$, and K is a symmetric body of $\mathbb{R}^m$, the class $W^p(K),\quad 1\leq p\leq\infty,$ coincides with the well-known class of entire functions of exponential type K whose traces on $\mathbb{R}^m$ belong to the space $L^p(\mathbb{R}^m)$ [7, 8]. For the case of compact G the functions from $W^p(K),\quad 1\leq p\leq\infty,$ are finite linear combinations of characters of G.

In papers [1, 2] it is proved that $W^p(K),\quad 1\leq p\leq\infty,$ is the shift invariant closed subspaces of $L^p(G)$. In the case when $1\leq p\leq 2$, the set $W^p(K)$ coincides with the set $F^p(K)$, which consists of continuous functions on G, whose Fourier transform supports belong to K. Moreover, the set of functions $W^p(K)$ for all possible compact $K\in U_{\hat{G}}$ is dense in $L^p(G),\ \ 1\leq p<\infty.$ In connection with the density problem we remark the following.

Let K be a symmetric compact set from $\hat{G}$. In [4] we have introduced a definition of B-property, which is an analogy of the notion of convexity for locally compact Abelian groups. According to this definition, a set K possesses the B-property if the element $g\in\hat{G}$, which admits for a certain natural number n the representation $g^n=g_1^{n_1}\cdots g_k^{n_k}$, where $n_1,\ \cdots\ ,n_k$ are natural numbers, $n=n_1+\cdots+n_k$, while $g_1,\cdots,g_k\in K,$ belongs to K. If T is a set from $U_{\hat{G}}$, without the B-property, it can be put in a $K\in U_{\hat{G}}$ which does have the B-property. To construct such a minimal set K, we must consider all elements $g\in\hat{G}$ which can be represented in a form $g^n=g_1^{n_1}\cdots g_k^{n_k}$, where $g_1,\cdots,g_k\in T,\quad n_1,\cdots\ ,n_k\in N,\quad n=n_1+\cdots+n_k$ and connect them to T such g, if $g\notin T.$ If we connect to T all such g, then the obtained set K possesses the required property. Really, let us consider elements $g_1,\cdots,g_k\in K$ and suppose that $g^n=g_1^{n_1}\cdots g_k^{n_k}$ for some $g\in G,\ \ g_1,\cdots,g_k\in K,$ $n_1,\cdots\ ,n_k\in N,\quad n=n_1+\cdots+n_k$. It suffices to consider the case k=2. Let $g_1^{n_1}=g_{11}^{n_{11}}\cdots g_{1h}^{n_{1h}}$, $g_2^{n_2}=g_{21}^{n_{21}}\cdots g_{2l}^{n_{2l}},\ \ n_1=n_{11}+\cdots+n_{1h},\ \ n_2=n_{21}+\cdots+n_{2l},\ \ g_{1i}\in T,\ \ i=1,\cdots,h,\ \ g_{2j}\in T,\ \ j=1,\cdots,l$. Let n be the least common multiple of the numbers n_1 and n_2, and $n=n_1m_1=n_2m_2$. Let $g^s=g_1^{s_1}g_2^{s_2},\ \ s=s_1+s_2\ \ (s_1,\ \ s_2\in N)$. Let us prove that $g\in K$. We have $g^{ns}=(g^s)^n=g_1^{ns_1}g_2^{ns_2}=$ $=g_{11}^{n_{11}m_1s_1}\cdots g_{1h}^{n_{1h}m_1s_1}g_{21}^{n_{21}m_2s_2}\cdots g_{2l}^{n_{2l}m_2s_2}$. Here $n_{11}m_1s_1+\cdots+n_{1h}m_1s_{12}+n_{21}m_2s_2+\cdots+n_{2l}m_2s_2=ns,$ $g_{11},\cdots,g_{2l}\in T$ and by construction of K, we have that $g\in K$. Thus K has the B-property. Inclusion $K\in U_{\hat{G}}$ is clear. Thus every $T\in U_{\hat{G}}$ may be put in some $K\in U_{\hat{G}}$, which has the B-property. It is shown in [1], that $W^p(T)\subset W^p(K),\ \ 1\leq p\leq\infty$, for every $T\subset K$, and therefore we can take the sets $W^p(K)$ as a dense set in $L^p(G)$ for all $K\in U_{\hat{G}}$, having the B-property too.

Let I be an ordered unbounded set $I \subset \mathbb{R}^+$ and consider a generalized sequence of sets $K \in U_{\hat{G}}$, such that $K_\alpha \subset K_\beta$ if $\alpha < \beta$ $(\alpha,\ \beta \in I)$ and $\bigcup_{\alpha\in I} K_\alpha = \hat{G}$. For a such sequence $L^p(R^m)$, $1 \le p \le \infty$ we have studied in [5] the following sequence of positive operators

$$\sigma_{K_\alpha}(f)(g) \equiv (f * V_{K_\alpha})(g) = \int_G f(h)\ \ V_{K_\alpha}(h^{-1}g)dh, \tag{2}$$

where

$$V_{K_\alpha}(g) = (\text{mes}K_\alpha)^{-1}((\hat{1})_{K_\alpha}(g))^2, \qquad K_\alpha \in U_{\hat{G}}. \tag{3}$$

The kernels (3) are mentioned in [6], (Ch.5, §1) and they represent the limiting case of the kernel $V_{K,T}$ defined by (1), when K converges to the unity of $\hat{G}$ and T is replaced by K_α. In [5] it is proved that if $f \in L^p(G)$, $1 \le p \le \infty$ then $\sigma_{K_\alpha}(f) \in W^p(K_\alpha^2)$.

Some results concerning positive linear operators and the approximation of continuous functions on locally compact Abelian groups are given in [9]. First of all the positive operators (2) are of importance as in the case $G = \mathbb{R}^m$ they coincide with the well-known Fejér operator. For example, if $m = 1$, $I = (0,\infty)$, $\alpha \in I$ and $K_\alpha = [-\alpha,\ \alpha]$, we have the sequence of operators (see, for example, [10], 3.1.2, p. 122)

$$\sigma_{[-\alpha,\alpha]}(f)(x) \equiv \sigma_\alpha(f)(x) = \frac{2}{\pi\tau}\int_{-\infty}^{\infty} f(t)(x-t)^{-2}\sin^2\frac{\tau}{2}(x-t)dt, \qquad \tau \in \mathbb{R}^+.$$

In the case when $G = \mathbb{R}^m$, $m \ge 1$ and K is the ball of a radius $\tau \in \mathbb{R}^+$, the kernel of the operator (2) takes the form

$$V_K(x) = (\omega_m)^{-1}\,|x|^{-m}\ J_{m/2}^2(\tau\,|x|/2), \qquad x \in \mathbb{R}^m,$$

where $J_{m/2}$ is the Bessel function of order $m/2$ and ω_m is the area of the m-dimensional unit sphere. In that case, the operator (2) is studied in [11] in connection with the saturation problem in $L^p(R^m)$, $1 \le p \le \infty$. In [5] it is proved that if $f \in L^p(G)$, $1 \le p < \infty$, and the sequence of sets $K_\alpha \in U_{\hat{G}}$ satisfies the condition

$$\lim_{\alpha\to\infty}\frac{\text{mes}\ \ (TK_\alpha)}{\text{mes}\ \ K_\alpha} = 1 \tag{4}$$

for all fixed $T \in U_{\hat{G}}$, then

$$\lim_{\alpha\to\infty}\|\,f - \sigma_{K_\alpha}(f)\,\|_{L^p(G)} = 0.$$

Under the condition (4) the sequence (2) converges to f also in the space $L^\infty(G)$, but in the weak* topology of $L^\infty(G)$. An analogous result is valid also in the space $M(G)$ of bounded regular complex valued Borel measures on G [5]. In addition to these results we state the following

Theorem 1. *Let* $\{K_\alpha\}$ *be a sequence in* $U_{\hat{G}}$ *satisfying (4) and* $S \subset G$ *is a compact set. If a function* $f \in L^\infty(G)$ *is continuous at a neighborhood of S, then* $\sigma_{K_\alpha}(f)$ *converges to f uniformly on S as* $\alpha \to \infty$.

Proof. Since f is uniformly continuous on S, given $\delta > 0$ find a neighborhood $V \in U_{\hat{G}}$ of the unity of

$\hat{G}$ such that

$$|f(hg)-f(g)|< \delta/2 \qquad \text{for all} \quad g\in T, \quad h\in V. \tag{5}$$

In [5] (Proposition 4) it is proved that under assumption (4) the sequence of kernels $\{V_{K_\alpha}\}$ represents an approximate unity. This means that for every fixed $U\in U_{\hat{G}}$ the following equalities hold

$$\lim_{\alpha\to\infty}\int_{G\setminus V} V_{K_\alpha}(g)dg=0, \qquad \int_G V_{K_\alpha}(g)dg=1.$$

Therefore there exists a $\alpha_0>0$ such that

$$\int_{G\setminus V} V_{K_\alpha}(g)dg < \frac{\delta}{4\|f\|_{L^\infty(G)}}, \quad \text{for all } \alpha>\alpha_0. \tag{6}$$

By use of the equality $\int_G V_{K_\alpha}(g)dg=1$ we have

Applying (5) and (6), and representing the integral of the right-hand as $\int_V + \int_{G\setminus V}$, we obtain

$$\sup_{g\in T}|f(g)-\sigma_{K_\alpha}(f)(g)|\le \delta/2+\delta/2=\delta, \qquad \text{when} \quad \alpha\to\infty.$$

Now we prove that the rate of convergence of the operators σ_{K_α} to the identical operator in general is not greater than $\text{mes}K_\alpha$.

Theorem 2 . *Let $f\in L^p(G)$, $1\le p\le 2$, and a sequence of sets $K_\alpha\in U_{\hat{G}}$ satisfy the condition*

$$\limsup_{\alpha\in I}\{\text{mes}\,K_\alpha\setminus\text{mes}\,(K_\alpha\cap(\chi K_\alpha))\}\neq 0 \tag{7}$$

for any fixed $\chi\in\hat{G}$. Then from the condition

$$\|f-\sigma_{K_\alpha}(f)\|_{L^p(G)}=o(\text{mes}\,K_\alpha), \qquad \alpha\to\infty, \tag{8}$$

it follows that $f(g)=0$ a.e. on G.

Proof. Since $f\in L^p(G)$, $1\le p\le 2$, and $V_{K_\alpha}\in L^1(G)$ for all $\alpha\in I$, we have for the Fourier transform of (2) ([12], Ch.8)

$$(\sigma_{K_\alpha}(f))^\wedge(\chi)=\hat{f}(\chi)(V_{K_\alpha})^\wedge(\chi), \tag{9}$$

and

$$(V_{K_\alpha})^\wedge(\chi)=(\text{mes}\,K_\alpha)^{-1}(((1)_{K_\alpha}*(1)_{K_\alpha})^\wedge)^\wedge(\chi).$$

It is clear that $(1)_{K_\alpha}*(1)_{K_\alpha}\in L^1(\hat{G})$ and $((1)_{K_\alpha}*(1)_{K_\alpha})^\wedge\in L^1(G)$. Therefore we have from (9)

$$(\sigma_{K_\alpha}(f))^\wedge(\chi)=\hat{f}(\chi)(\text{mes}\,K_\alpha)^{-1}((1)_{K_\alpha}*(1)_{K_\alpha})(\chi^{-1})$$

It follows from here

$$(\text{mes } K_\alpha)(f-\sigma_{K_\alpha}(f))^\wedge(\chi)=\hat{f}(\chi)(\text{mes } K_\alpha - \int_{K_\alpha}(1)_{K_\alpha}(h^{-1}\chi^{-1})dh). \tag{10}$$

The upper bound of the L^q norm $(p^{-1}+q^{-1}=1)$ on the left-hand is, according to the Hausdorf inequality ([12], §31.22), $(\text{mes } K_\alpha)\|f-\sigma_{K_\alpha}\|_{L^p(G)}$. Then it follows from (8) and (10)

$$\lim_{\alpha\to\infty}\|\hat{f}(\chi)(\text{mes } K_\alpha - \int_{K_\alpha}(1)_{K_\alpha}(h^{-1}\chi^{-1})dh)\|_{L^q(G)}=0.$$

According to (3), this means that $\|\hat{f}\|_{L^q(G)}=0$, i.e. $f(g)=0$ a.e. on G ([12], §31.31).

Now we give examples of operators σ_{K_α} for some groups G and sets K_α:

1. Let $G=\mathbb{R}^m$, $m\ge 1$ and consider the sets $K_\alpha=\{u:\ u\in\mathbb{R}^m,\ d(u)\le\alpha\},\ \alpha>0$, where $d^2(u)=(u,Au)=$ $=\sum_{k,l=1}^m a_{k,l}u_k u_l$ and $A=(a_{k,l})_{k,l=1}^m$ is a positive definite matrix. It is clear that $K_\alpha\subset U_{\hat{G}}=U_{R^m}$. It is possible to calculate $\hat{(1)}_{K_\alpha}$ by analogy to [13] (we must keep in mind that according to our agreement the Haar measure μ in $\mathbb{R}^m$ is normalized in such a way that $\mu(E)=(2\pi)^{m/2}l(E),\quad E\subset\mathbb{R}^m$, where $l(E)$ is the Lebesgue measure of E). Actually, we can obtain the following representation of σ_{K_α}

$$\sigma_{K_\alpha}(F)(x)=\frac{2^m\pi^{m/2}\Gamma(m/2+1)}{\sqrt{\det\Lambda}}\int_{\mathbb{R}^m}f(t+x)\frac{J^2_{m/2}((\alpha/2)\sqrt{(x,A^{-1}x)})}{(x,A^{-1}x)^{m/2}}dx, \tag{11}$$

where $J_{m/2}$ is the Bessel function of order $m/2$ and Γ is the Euler gamma function.

Let us consider the matrix B, whose columns are orthonormal eigenvectors of A. Let B' be the matrix conjugate to B and Λ be the diagonal matrix composed with the eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_m$ of A. It is well-known that $\Lambda=B'AB$, $\det A=\det\Lambda$, $\det B=1$. The change of variables $u=Bv$, $v_k=\lambda_k^{-1/2}z_k$ gives

$$(u,Au)=(Bv,ABv)=(v,B'ABv)=(v,\Lambda v)=|z|^2=z_1^2+z_2^2+\cdots+z_m^2.$$

Thus $d(u)\le\alpha\ \Leftrightarrow\ |z|\le\alpha$. This linear correspondence between K_α and $C_\alpha:=\{z\in R^m\ :\ |z|\le\alpha\}$ is $z=(B\Lambda)^{-1}u$. By this map the image of $C_\alpha+\mu$ for any $\mu\in K_\alpha$ is translation of C_α by the vector $\mu'=(B\Lambda)^{-1}u$. It is clear that $\text{mes } K_\alpha\backslash\text{mes }(K_\alpha+\mu)=\text{mes } C_\alpha\backslash\text{mes }(C_\alpha+\mu')$. It may be proved by calculation that $\text{mes } C_\alpha\backslash\text{mes }(C_\alpha+\mu')\ge C(\mu')\alpha^{m-1}$, where $C(\mu')$ depends only on μ' a positive number. Thus, in the considered example condition (7) is satisfied. Therefore Theorem 2 implies that in this example the approximation by operators (11) of order α of nontrivial functions from $L^p(\mathbb{R}^m)$, $1\le p\le 2$, is impossible.

2. If $G=\mathbb{Z}$ is the group of whole numbers, then $\hat{G}=E$ is the unit circumference from $\mathbb{R}^2$ up to an isomorphism. Characters of G have the form $\chi_t(n)=t^n,\ n\in Z,\ t\in E$. Let $K_\alpha=\{e^{i\theta}:-\pi+\frac{\pi}{\alpha}\le\theta\le\pi+\frac{\pi}{\alpha},\ \ 1<\alpha<\infty\}$. Take, as a dual measure on E, the arc length divided into $i\sqrt{2\pi}$. We have

$$(\hat{1})_{K_\alpha}(n) = \frac{1}{i\sqrt{2\pi}} \int_{K_\alpha} \xi^n d\xi = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{\pi}\ \ (n+1)} \sin \dfrac{(\alpha-1)(n+1)\pi}{\alpha}, & n \neq -1, \\ \sqrt{2\pi}\ \ \dfrac{\alpha-1}{\alpha}, & n = -1, \end{cases}$$

and therefore, in this case

$$\sigma_{K_\alpha}(f)(n) = \frac{\alpha}{(\alpha-1)\pi^2} \sum_{k=-\infty,\ k\neq n+1}^{\infty} f(k) \frac{\sin^2 (\alpha-1)(n-k+1)\pi/\alpha}{(n-k+1)^2} + \frac{\alpha}{\pi} f(n+1).$$

3. Let $G = \mathbb{R}^+$ be the multiplicative group of positive integers with the unit e=1. This group has a character $\chi(\xi) = \xi^{ix}$, where $x \in \mathbb{R}$ and $\hat{G} = \mathbb{R}$ up to an isomorphism. Take as K_α the interval $K_\alpha = [-\alpha,\ \alpha],\quad \alpha \in \mathbb{R}^+$. Then

$$(\hat{1})_{K_\alpha}(\xi) = \int_{-\alpha}^{\alpha} \xi^{ix} dx = \frac{2\ \sin\ (\alpha\ \ln\ \xi)}{\ln\ \xi}$$

And, therefore, in this case

$$\sigma_{K_\alpha}(f)(x) = \frac{1}{\alpha\pi} \int_0^\infty f(h) \frac{\sin^2(\alpha \ln \frac{x}{h})}{h \ln^2 \frac{x}{h}} dh.$$

4. Let $G = \mathbb{Q}_p$ be the field of the p-adic numbers with a prime p. With respect to the additional operation of p-adic numbers, $\mathbb{Q}_p$ is a locally compact Abelian group [14]. Its dual group is isomorphic to the addition group $\mathbb{Q}_p$. The character χ_ξ, corresponding to a p-adic number ξ, has the form $\chi_\xi(x) = \exp(2\pi i \{\xi x\}_p)$. Here $\{x\}_p$ is defined by the p-adic expansion $x = \sum_{n \geq \text{ord}_p(x)} a_n p^n$ as

$$\{x\}_p = \begin{cases} \sum_{n=\text{ord}_p(x)}^{-1} a_n p^n, & \text{if} \quad \text{ord}_p(x) < 0, \\ 0, & \text{if} \quad \text{ord}_p(x) \geq 0, \end{cases}$$

and ξx is the product of p-adic numbers ξ and x in the field $\mathbb{Q}_p$.

Let $n \in \mathbb{Z}$, $\alpha = p^n$ and K_α be the p-adic ball of the radius p^n with the center in zero, i.e. $K_\alpha = \{h \in \mathbb{Q}_p\ :\ |h|_p \leq p^n\}$, where $|h|_p$ is the p-adic norm of h. The Haar measure of this ball is mes $K_\alpha = p^n$ ([14], §4, (2.3)). By means of the formula ([14], §4, (3.1))

$$\int_{K_\alpha} \chi_\xi(x) dx = \begin{cases} p^n, & \text{if} \quad |\xi|_p \leq p^{-n}, \\ 0, & \text{if} \quad |\xi|_p > p^{-n+1}, \end{cases}$$

we obtain that in the considered case the operator σ_{K_α} has the form $\sigma_{K_\alpha}(x) =$

$$= p^{-n} \int_{\mathbb{Q}_p} f(\xi + x) d\xi \left(\int_{|t|_p \leq p^n} e^{2\pi\, i\, \{t\, \xi\}_p} dt \right)^2 = p^n \int_{|\xi|_p \leq p^{-n}} f(\xi + x) d\xi\ .$$

მათემატიკა

ფეიერის განზოგადებული ინტეგრალის აპროქსიმაციული თვისებების შესახებ

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შესწავლილია ლოკალურად კომპაქტურ აბელის ჯგუფზე ჰაარის ზომით რომელიღაც $p \geq 1$-სათვის p-ინტეგრებად და აგრეთვე არსებითად შემოსაზღვრულ ფუნქციათა აპროქსიმაციის საკითხები. მააპროქსიმირებელ აგრეგატებად აღებულია ფეიერის ცნობილი დადებითი ოპერატორების ანალოგები.

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