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Mathematics

Maxwell Conjecture and Polygonal Linkages

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ABSTRACT. Motivated by the famous Maxwell conjecture on the number of equilibria of point charges, we discuss the electrostatic potential of point charges placed at the vertices of polygonal linkage. In particular, we establish that electrostatic potential is a Morse function on the moduli space of a generic polygonal linkage, which yields estimates for the number of its equilibria. For quadrilateral linkage, we present a number of results on the structure of electrostatic equilibria and, as a by-product, prove that its shape is completely controllable by changing the charge at just one vertex. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *point charge, electrostatic potential, equilibrium, polygonal linkage, moduli space, Morse function, critical point.*

1. The famous conjecture of J.C.Maxwell states that the number of equilibria of n equal point charges in $\boldsymbol{R}^3$ does not exceed $(n-1)^2$ [1]. This conjecture remains unproven even for n=3 and the best established estimate for n=3 is 12 [2]. Being classical and intriguing, this conjecture gained a lot of attention of researchers (see, e.g., [2]). In particular, various modifications and special cases of Maxwell conjecture have been considered, which led to a number of interesting mathematical results obtained by topological (see, e.g., [3]) and analytical methods (cf. [4]).

Along these lines, we suggest an analog of Maxwell conjecture in the context of polygonal linkages, present several related results for linkages with a small number of vertices and discuss a few promising topics naturally arising in the framework of our setting. Our approach relies on the topological results on moduli spaces of linkages [5] and signature formulae for the topological invariants of functions on moduli spaces [6].

2. Let us first recall some definitions and facts concerned with electrostatic potentials. By definition, the *electrostatic (Coulomb) potential* of a system V of unit charges q=1 placed at the points $v_i \in \boldsymbol{R}^3$ is a rational function on $\boldsymbol{R}^3$ defined by the formula $\Psi_V(P) = \Sigma\, (d(P,v_i))^{-1}$, where $P \in \boldsymbol{R}^3$ and $d(P,v_i)$ denotes the Euclidean distance between P and v_i. The *electrostatic energy* of V is defined by the formula $E_s(V) = \Sigma\, (d(v_i,v_j))^{-1}$, where the sum is taken over all pairs of nonequal indices.

We will also consider the planar analog of G called *planar electrostatic potential* Φ which is defined for a system V of unit charges placed at the points $z_i \in \mathbb{R}^2$ by the formula $\Phi_V(z) = \Sigma \operatorname{Ln}(d(z,z_i))$, where Ln denotes the natural logarithm and $z \in \mathbb{R}^2$. The *planar electrostatic energy* of V is defined as $E_p(V) = \Sigma \operatorname{Ln}(d(z_i, z_j))$, where the sum is taken over all pairs of nonequal indices.

For a system V of points $z_i \in \mathbb{R}^2$ we also introduce a polynomial $P_V(z) = \Pi (z - z_i)$ having the points z_i as its zeroes.

Notice that the natural planar analog of Maxwell conjecture holds true in virtue of a classical observation of C.Gauss. Indeed, Gauss showed that the critical points of Φ_V coincide with the zeroes of derivative $(P_V)'$ of polynomial P_V (cf. [4], Ch.1). This immediately yields the exact estimate for the number of equilibria of planar electrostatic potential.

Fact 1. The number of critical points of planar electrostatic potential of n unit charges in the plane does not exceed $n - 1$.

For n=3, using this observation of Gauss and a well-known theorem on zeroes of derivatives of cubic polynomial ([4], Ch.1) one obtains an explicit geometric description of equilibria of planar electrostatic potential.

Fact 2. The critical points of planar electrostatic potential of three unit charges placed at the vertices of triangle Δ lie at the focuses of the ellipse tangent to the sides of Δ at their midpoints.

Such an ellipse always exists and is often called the *Steiner ellipse* of Δ [7]. It is known that it has the maximal area among all ellipses contained in Δ, which suggests that the equilibria of potentials might be connected with certain extremal properties of arising configurations of equilibria. Both these facts serve as paradigms for our considerations.

3. We present now some definitions and facts concerned with polygonal linkages. Recall that a polygonal linkage (or a closed polygonal k-chain) L is defined by a k-tuple of positive numbers l_i called *sidelengths* of L. In the case of a closed polygonal chain it is always assumed that each of the sidelengths is not greater than the sum of all other ones [5]. A polygonal linkage is called *regular* if all sidelengths are equal. The *planar configuration space* C(L) of a polygonal k-chain L is defined as the collection of all k-tuples of points v_i in Euclidean plane such that the distance between v_i and v_{i+1} is equal to l_i, where it is assumed that $v_{k+1} = v_1$. Each such collection of points is called a *configuration* of L. A configuration is called *convex* if the corresponding polygon is convex. Factoring C(L) over the natural diagonal action of SO(2) one obtains the *(planar) moduli space* $M(L) = M_2(L)$ [5]. The subset of M(L) formed by the convex configurations will be denoted $M^c(L)$. Analogously, one can define higher configuration spaces $M_N(L)$ of which we will only consider $M_3(L)$ and denote it S(L). Moduli spaces, as well as configuration spaces, are endowed with natural topologies induced by Euclidean metric.

It is easy to see that the planar moduli space can be identified with the subset of configurations such that $v_1 = (0,0)$, $v_2 = (l_1,0)$. It is well known that, for a closed k-chain, the moduli space has a natural structure of compact orientable real-algebraic set of dimension $k - 3$. Let us say that a polygonal linkage is *degenerate* if it has an *aligned configuration*, i.e., a configuration where all vertices lie on the same straight line. It is well known that this happens if and only if there exists a k-tuple of "signs" $s_i = \pm 1$ such that $\Sigma s_i l_i = 0$. The moduli spaces $M_N(L)$ of polygonal linkage L is smooth (does not have singular points) if and only if L is nondegenerate (see, e.g., [5]).

4. We are now able to describe the settings discussed in the sequel. The main idea is to fix a polygonal linkage L as above and consider E_p and E_s as functions on the corresponding moduli spaces M(L) and S(L) respectively. We present now a rigorous description of this setting in the case of *quadrilateral linkage*.

So let Q = Q(a, b, c, d) be a nondegenerate quadrilateral linkage with pairwise non-equal lengths of the sides. For brevity, such a linkage will be called *admissible*. For each configuration V of Q, let us consider its energies $E_p(V)$ and $E_s(V)$. Since Q does not have configurations with coinciding vertices, both these energies are smooth (infinitely differentiable) functions on M(Q) and S(Q) respectively. So one may consider their critical points in these moduli spaces which in fact correspond to the *physical equilibria* of linkage Q subject only to electrostatic forces between its vertices.

We can now formulate the two problems we are interested in: (P1) find the number of equilibria of E_p and E_s for a given linkage, and (P2) find the maximal possible number of equilibria of E_p and E_s over the set of all admissible quadrilateral linkages.

Notice a conceptual analogy of (P2) with Maxwell conjecture. However, an essential **difference** is that here we consider the **equilibria of the linkage itself** and not the equilibria of its electrostatic potential in the ambient space. In such a setting the problem acquires several new aspects, which lead to the following results in the spirit of [6]. First of all, problem (P1) can be solved using our general approach based on signature formulae for topological invariants [6].

Theorem 1. *For an admissible quadrilateral linkage* Q*, the number of equilibria of any of energies* E_p *and* E_s *can be calculated as the signature of a quadratic form with the coefficients algebraically expressible through the sidelengths of* Q.

Outline of the proof. The result follows by applying the signature formula for Euler characteristic to the polynomial system for the equilibria obtained by the method of Lagrange multipliers. We give first an outline of the proof in the case of planar potential E_p.

The main idea in this case is to simplify the expression for E_p by considering the length of diagonals of configuration V as coordinates on M(L). More precisely, we put $x = d(v_1, v_3)$, $y = d(v_2, v_4)$ and notice that the pair (x,y) completely determines the shape of configuration V, i.e. its class in M(Q). In these coordinates one has: $E_p = x^{-1} + y^{-1} + C$, where C is the constant equal to the sum of the terms corresponding to the sides of V. Obviously, C has no influence on the critical points of E_p.

Now, from a classical result known as the *Euler four points formula* [7] follows that the moduli space M(Q) is defined by the following polynomial equation in coordinates (x,y):

$$x^2y^4 + y^2x^4 - (a^2 + b^2 + c^2 + d^2)\, x^2y^2 + (a^2 - d^2)(b^2 - c^2)\, x^2 + (a^2 - b^2)(d^2 - c^2)\, y^2 + C_1 = 0, \quad (*)$$

where $C_1 = (b^2d^2 - a^2c^2)(b^2 + d^2 - a^2 - c^2)$ is a constant.

Denoting by T the left hand side of this equation we see that point V is critical for E_p if and only if the gradients of E_p and T with respect to (x,y) coordinates are proportional at this point which gives another polynomial equation $T_1 = 0$. In this way we obtain a (2x2)-system of polynomial equations and it is easy to verify that its jacobian is not identically equal to zero. This means that one can express the number of real solutions to this system by the signature formula [6], which yields the desired result.

In case of E_s the proof is in a sense easier since the lengths of diagonals (x,y) are independent local coordinates on S(Q) in the interior of a certain box $B = [x_m, x_M] \times [y_m, y_M]$ defined by the regions of values of x and y on S(Q). This follows from the existence of diagonal bendings [5] which change one of the diagonals leaving the second one unchanged. So the equilibria are given by counting the zeroes of gradient E_s inside the box B and checking the degenerate configurations of Q with extremal values of x or y. Both these procedures can be done effectively using signature formulae and the result follows.

The above considerations and methods used in [6], [8] enable one to show that the equilibria are in fact nondegenerate in the sense of Morse theory.

Theorem 2. *For a generic admissible quadrilateral linkage* Q, E_p *and* E_s *are Morse functions on* M(Q) *and* S(Q) *respectively.*

The proof is obtained by analyzing the constrained extremal problem for the potential in question. To this end we consider the extended Hessian matrix of Lagrange function in coordinates (x,y) introduced above. Due to the simple form of E_p and E_s in these coordinates, the determinant of Hessian T_2 can be computed explicitly. We compute next the resultant of polynomials T, T_1, T_2 and verify that it does not vanish identically, which implies that the zeroes of the determinant generically cannot coincide with the solutions to the Lagrange system for equilibria.

Results of such kind are useful because they enable one to estimate the number of equilibria using Morse inequalities and similar topological tools [3], [9]. For quadrilateral linkages, this is not indeed interesting since both M(L) and S(L) have very simple topology but this idea may yield a useful paradigm in the general case of linkage with arbitrary number of sides $n > 4$.

After having found the polynomial system real solutions to which give the equilibria of potential in moduli space one can try to describe the bifurcation diagram of this system in the space of parameters (a, b, c, d) and find the number of equilibria in each component of its complement by the aforementioned signature formulas. We were only able to do this for the planar potential E_p which yielded a solution to problem (P2) in this case.

Theorem 3. *For any admissible linkage* Q, E_p *has no more than 8 critical points on* M(Q).

A detailed proof of this result will be published elsewhere. Analogous results for linkages with the number of sides bigger than four would be difficult to prove by the same method since the arising polynomial systems involve not less than three variables in which cases computers are usually unable to calculate the result using signature formulae.

5. The above results on electrostatic equilibria of linkages have the following curious application in the spirit of control theory. Consider an admissible quadrilateral linkage Q as above and place positive charges at its vertices. Among the critical points of E_p and E_s the global minima are especially important since they give the *stable equilibria* of the linkage subject to only electrostatic forces.

It is easy to see that the global minimum of E_s can only be attained at a planar configuration of quadrilateral Q. Indeed, for each nonplanar configuration one can increase the length of at least one of the diagonals without changing another one which clearly decreases the value of E_s. Thus in this context it is sufficient to consider both potentials as functions on M(Q).

As is geometrically obvious and can be easily verified, given a non-convex planar configuration, one can increase both of its diagonals simultaneously by deforming the linkage [7]. Thus the global minima of both potentials always belong to $M^c(Q)$. In fact, one can prove a stronger result.

Theorem 4. *For an admissible quadrilateral linkage* Q *with arbitrary positive charges at its vertices, the global minimum of* E_s *on* M(Q) *is unique, and the same is true for* E_p.

This result suggests the idea of controlling the shape of Q by changing the charge at one of the vertices of Q. More precisely, we fix the positions of the first two vertices of Q, place unit positive charges at all vertices except the first one placed at the origin, and permit ourselves to change the value $q > 0$ of charge at the first vertex. By Theorem 4, for each $q > 0$ we have a single stable equilibrium of E_s, i.e. a well-defined point in $M^c(Q)$ which we denote by $V(q)$. A natural question now is whether this mapping is surjective as a mapping from $\mathbf{R}_+$ into $M^c(Q)$. A positive answer to this question would mean that we may force the linkage to take any convex shape from $M^c(Q)$ by choosing a proper value of q.

Theorem 5. *The mapping from* $\mathbf{R}_+$ *into* $M^c(Q)$ *defined by sending* q *to* $V(q)$ *is surjective on the interior of* $M^c(Q)$.

Outline of the proof. First of all, using Lagrange method it is easy to see that the configuration with the lengths of diagonals equal to (x_o, y_o) is the global minimum of E_s when the charge at the first vertex is equal to

$$q = (x_o)^2 (y_o)^{-2} T_x (T_y)^{-1}, \qquad (**)$$

where the subscripts denote partial derivatives and both y_o and T_y are non-zero.

If $y_0 = 0$ one has an analogous relation with the roles of x and y exchanged. It only remains to show that the obtained value of q is indeed positive. This can be proven by analyzing the implicit functions of the form y(x) and x(y) obtained from equation (*). A simple argument implies that in convex position both partial derivatives have the same sign, which completes the proof.

Using the ideology and terminology of control theory [10] this result means that convex configurations of charged quadrilateral Q as above can be completely controlled by the value of charge at just one of its vertices. The same result holds for E_p and its proof can be obtained by a slight modification of the relation (**) in the above argument. In this context the result for E_s is more important since in real life one always deals with the Coulomb potential

It is now obvious that similar problems make sense for linkages with arbitrary number of sides. Discussion of arising generalizations and applications will be continued in forthcoming publications of the author.

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მაქსველის ჰიპოთეზა და სახსრული მრავალკუთხედები

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