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Mathematics

On Analytic and Harmonic Functions with a Dirichlet Finite Integral in the Unit Circle

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ABSTRACT. The paper deals with the study of integral representations and boundary properties of harmonic functions having an analytical and summable gradient with a summable derivative in the unit circle. One theorem of S.M. Nikolski on the existence of a boundary value of a harmonic function is formulated more precisely. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *analytic function, majorant function, gradient, Bergman space.*

Denote by D the open unit circle on the $\mathbb{C}$-complex plane, and by T the unit circumference with centre at the origin, i.e.

$$D=\{z\in\mathbb{C}:\ |z|<1\},$$
$$T=\{t\in\mathbb{C}:\ |t|=1\}.$$

Assume that $H(D)$ is a set of all analytical functions in D. Let $p>0$ be some fixed number. Denote by A_p' the space of all those functions $f\in H(D)$ which satisfy the condition

$$\int_D |f'(z)|^p \, d\sigma(z) < +\infty, \tag{1}$$

where $d\sigma(z)=dx\,dy$ is a usual plane Lebesgue measure in D and $z=x+iy,\quad x,y\in R=(-\infty,+\infty)$.

If $p = 2$, then the space A_2' is called the Dirichlet space or the space of analytic functions with a finite integral (see [1]).

Denote by $H_p'(D)$ the Bergman space of analytic functions in the circle D, i.e.

$$H_p'=\left\{f\in H(D):\int_D |f(z)|^p \, d\sigma(z)<\infty\right\}.$$

Assume that $f(z)=u(z)+iv(z)$, where $u=\operatorname{Re} f$ and $v=\operatorname{Im} f$, then since $\forall z\in D$

$$f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x},$$

we have

$$|grad\,u|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}=\sqrt{\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2}=|grad\,v|=|f'(z)|.$$

This implies

$$f\in A_p' \Leftrightarrow \int_D |grad\,u|^p\, d\sigma(z)<\infty,$$

$$f\in A_p' \Leftrightarrow \int_D |grad\,v|^p\, d\sigma(z)<\infty.$$

The theorems below are valid.

Theorem 1. $f\in A_1'(D)$ *if and only if the equality*

$$f(z)=f(0)+\frac{z}{\pi}\int_D\frac{f'(t)d\sigma(t)}{1-z\bar{t}} \tag{2}$$

holds $\forall z\in D$.

Proof. Since

$$\int_D |f'(z)|\,d\sigma(z)<\infty,$$

we have $f'\in H_1'(D)$. According to Kabaila's theorem (see [2]), $\forall z\in D$ we obtain

$$f'(z)=\frac{1}{\pi}\int_D\frac{f'(t)d\sigma(t)}{\left(1-z\bar{t}\right)^2} \Leftrightarrow f(z)=f(0)+\int_0^z f'(\xi)d\xi=f(0)+\frac{1}{\pi}\int_0^z\left[\int_D\frac{f'(t)d\sigma(t)}{\left(1-\xi\bar{t}\right)^2}\right]d\xi=$$

$$=f(0)+\frac{1}{\pi}\int_D f'(t)d\sigma(t)\cdot\int_0^z\frac{d\xi}{\left(1-\xi\bar{t}\right)^2}=f(0)+\frac{1}{\pi}\int_D f'(t)d\sigma(t)\frac{1}{\bar{t}}\left[\frac{1}{1-z\bar{t}}-1\right]=f(0)+\frac{z}{\pi}\int_D\frac{f'(t)d\sigma(t)}{1-z\bar{t}}.$$

The theorem is proved.

Theorem 2. *If* $f\in A_1'(D)$, *then* $\forall z\in D$

$$\frac{1}{\pi}\int_D\frac{\overline{f'(t)}d\sigma(t)}{1-z\bar{t}}=\overline{f'(0)}.$$

Proof. Indeed, assume $C(z,t)=\left(1-z\bar{t}\right)^{-1}$ and

$$C[f'](z)=\frac{1}{\pi}\int_D C(z,t)f'(t)d\sigma(t).$$

It is obvious that $C(z,t)=\sum_{n=0}^{\infty} z^n \overline{t}^n$. This series converges uniformly to t $\overline{D}$ because $|z|<1$. Also, using the function f' we have

$$\frac{1}{\pi}\int_D f'(t)\,d\sigma(t)=f'(0)$$

and $\forall n \in N$

$$\int_D f'(t)t^n\,d\sigma(t)=0 .$$

Therefore

$$C\left[\overline{f'}\right](z)=\frac{1}{\pi}\int_D \overline{f'}(z)C(z,t)\,d\sigma(t)=\frac{1}{\pi}\overline{\int_D f'(t)\overline{C(z,t)}\,d\sigma(t)}=\frac{1}{\pi}\overline{\int_D f'(t)\left[\sum_{n=0}^{\infty} z^n \overline{t^n}\right]d\sigma(t)}=$$

$$=\frac{1}{\pi}\overline{\int_D f'(t)\,d\sigma(t)}+\frac{1}{\pi}\sum_{n=1}^{\infty} z^n \overline{\int_D f'(t)t^n\,d\sigma(t)}=\overline{f'(0)}+0=\overline{f'(0)} .$$

The theorem is proved.

Theorem 3. *If* $f \in A_1'(D)$ *and* $f(0)=0$, *then* $\forall z \in D$

$$f(z)=\frac{z}{\pi}\int_D \frac{1+z\overline{t}}{1-z\overline{t}}u(t)\,d\sigma(t),$$

where $u=\operatorname{Re} f'$.

Proof. It is obvious that

$$\frac{1+z\overline{t}}{1-z\overline{t}}=2C(z,t)-1=\frac{2}{1-z\overline{t}}-1 .$$

Since $f' \in H_1'(D)$, we have (see [3])

$$f'(z)=\frac{1}{\pi}\int_D \operatorname{Re} f'(t)\left[\frac{2}{\left(1-z\overline{t}\right)^2}-1\right]d\sigma(t) \iff$$

$$\iff f(z)=\frac{1}{\pi}\int_0^z\left[\int_D u(t)\left(\frac{2}{\left(1-\xi\overline{t}\right)^2}-1\right)d\sigma(t)\right]d\xi=$$

$$=\frac{1}{\pi}\int_D u(t)\,d\sigma(t)\int_0^z\left[\frac{2}{\left(1-\xi\overline{t}\right)^2}-1\right]d\xi=\frac{1}{\pi}\int_D\left[\frac{2z}{1-z\overline{t}}-z\right]u(t)\,d\sigma(t)=\frac{z}{\pi}\int_D\frac{1+z\overline{t}}{1-z\overline{t}}u(t)\,d\sigma(t) .$$

The theorem is proved.

Theorem 4. *The following propositions are equivalent:*

1) $f \in A_1'(D)$,

2) $\forall z \in D,\ f(z) = \frac{z}{\pi}\int_D f'(t)C(z,t)\,d\sigma(t)$,

3) $f(z) = \frac{z}{\pi}\int_D f'(t)\left[2\operatorname{Re}C(z,t) - 1\right]d\sigma(t)$.

Proof. Theorem 1 implies that 1) $\Leftrightarrow$ 2). Let us show that 2) $\Rightarrow$ 3). Indeed, using Theorem 2 we obtain

$$\frac{z}{\pi}\int_D f'(t)\left[2\operatorname{Re}C(z,t)-1\right]d\sigma(t) = \frac{z}{\pi}\int_D 2\operatorname{Re}C(z,t)f'(t)\,d\sigma(t) - \frac{z}{\pi}\int_D f'(t)\,d\sigma(t) =$$

$$= \frac{z}{\pi}\int_D\left[C(z,t)+\overline{C}(z,t)\right]f'(t)\,d\sigma(t) - \frac{z}{\pi}\int_D f'(t)\,d\sigma(t) = \frac{z}{\pi}\int_D C(z,t)f'(t)\,d\sigma(t) +$$

$$+\frac{z}{\pi}\int_D \overline{C}(z,t)f'(t)\,d\sigma(t) = f(z) - \frac{z}{\pi}\int_D f'(t)\,d\sigma(t) +$$

$$+\frac{z}{\pi}\int_D f'(t)\,d\sigma(t) + \frac{z}{\pi}\sum_{n=1}^{\infty} z^n \int_D f'(t)t^n\,d\sigma(t) = f(z) + 0 = f(z).$$

If

$$f(z) = \frac{z}{\pi}\int_D f'(t)\left[2\operatorname{Re}C(z,t)-1\right]d\sigma(t),$$

then we have

$$f'(0) = \frac{1}{\pi}\int_D f'(t)\,d\sigma(t) \quad \text{as } z \to 0.$$

Hence we conclude that $f \in A_1'(D)$, i.e. 3) $\Rightarrow$ 1). The theorem is proved.

Assume $f \in L^1(D)$, where $L^1(D)$ is the Lebesgue space, and consider the operator

$$C[f](z) = \frac{1}{\pi}\int_D \frac{f(t)\,d\sigma(t)}{1 - z\bar{t}}.$$

Theorem 5. *If $f \in L^1(D)$, then the function $F : D \to \mathbb{C}$ defined $\forall z \in D$ by the equality*

$$F(z) = C[f](z) \tag{3}$$

belongs to the Hardy-Riesz class $H^p(D)$ $\forall p \in (0,1)$.

Proof. It is clear that *F* is an analytic function in *D*. We can assume without loss of generality that *f* is a real-valued non-negative function. Let us separate the real part of function (3). For this, we write the variables *z* and *t* in the explicit form: $z = re^{i\varphi}$, $t = \rho e^{i\theta}$, where $0 < r < 1$, $0 < \rho < 1$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le 2\pi$. Thus

$$F(z)=\frac{1}{\pi}\int_D \frac{f(\rho e^{i\theta})\rho\, d\rho\, d\theta}{1-r\rho e^{i(\varphi-\theta)}}=\frac{1}{\pi}\int_0^1\int_0^{2\pi}\frac{\left(1-r\rho e^{i(\theta-\varphi)}\right)f(\rho e^{i\theta})\rho\, d\rho\, d\theta}{\left|1-r\rho e^{i(\varphi-\theta)}\right|^2}=$$

$$=\frac{1}{\pi}\int_0^1\int_0^{2\pi}\frac{\left(1-r\rho\cos(\theta-\varphi)\right)f(\rho e^{i\theta})\rho\, d\rho\, d\theta}{1+r^2\rho^2-2r\rho\cos(\theta-\varphi)}+\frac{i}{\pi}\int_0^1\int_0^{2\pi}\frac{r\rho\sin(\varphi-\theta)f(\rho e^{i\theta})\rho\, d\rho\, d\theta}{1+r^2\rho^2-2r\rho\cos(\theta-\varphi)},$$

whence we obtain

$$u(z)=\operatorname{Re}F(z)=\frac{1}{\pi}\int_0^1\int_0^{2\pi}\frac{\left(1-r\rho\cos(\theta-\varphi)\right)f(\rho e^{i\theta})\rho\, d\rho\, d\theta}{1+r^2\rho^2-2r\rho\cos(\theta-\varphi)}\geq$$

$$\geq\frac{1}{\pi}\int_0^1\int_0^{2\pi}\frac{(1-r\rho)f(\rho e^{i\theta})\rho\, d\rho\, d\theta}{(1+r\rho)^2}\geq\frac{1-r}{\pi(1+r)^2}\int_0^1\int_0^{2\pi}f(\rho e^{i\theta})\rho\, d\rho\, d\theta=\frac{1-r}{\pi(1+r)^2}\|f\|\geq 0.$$

Therefore since the real part of the function F is non-negative, by virtue of Smirnov's theorem (see [4]),

$$F\in H^p(D),\quad \forall p\in(0,1).$$

Theorem 6. *If* $f\in A_1'(D)$*, then*

$$f\in\bigcap_{0<p<1}H^p(D).$$

Proof. According to Theorem 1, $\forall z\in D$ we have

$$f(z)=f(0)+\frac{z}{\pi}\int_D\frac{f'(t)\,d\sigma(t)}{1-z\bar{t}},$$

thus, by virtue of Theorem 5, $f\in H^p(D),\ \forall p\in(0,1)$, i.e. $A_1'(D)\subset H^p(D),\ \forall p\in(0,1)$.

Corollary 1. *If* $f\in A_1'(D)$*, then there exists an angular limit of the function* f

$$f^*\in L^p(T),\ \forall p\in(0,1)$$

a.e. on the circumference T.

Corollary 2. *If* $f\in A_2'(D)$*, where* $A_2'(D)$ *is the Dirichlet space, i.e.*

$$\int_D|f'(t)|^2\,d\sigma(t)<\infty,$$

then the function f *has an angular limit*

$$f^*(t)=\lim_{z\xrightarrow{\wedge}t}f(z)\ \text{and}\ f^*\in L^p(D),\quad \forall p\in(0,1)$$

a.e. on T.

S. M. Nikolski (see [5]) showed that if the harmonic function $u:D\to R$ satisfies the conditions:

1) $\int_D|u(z)|\,d\sigma(z)<\infty$,

2) $\int_D (grad\, u)^2\, d\sigma(z) < \infty$,

then there exists a radial limit of the function u

$$f(\theta) = \lim_{r\to 1} u\left(re^{i\theta}\right)$$

a.e. on the circumference T and $f \in L^2(T)$.

Let us assume that $f(z) = u(z) + iv(z)$, where v is a continuous conjugate harmonic function of u. Then

$$|f'(z)|^2 = (grad\, u)^2 = (grad\, v)^2,$$

and therefore, by condition 2), the function f has a.e. on T the angular boundary values $f^*(t)$ and $f^* \in L^p(T)$, $\forall p \in (0,1)$. Hence we conclude that the following statement is true.

Corollary 1. If the harmonic function $u : D \to R$ satisfies the condition

$$|f'(z)|^2 = (grad\, u)^2 = (grad\, v)^2,$$

then the function u has a.e. on the circumference T the angular limit $u^*\left(e^{i\theta}\right) = \lim_{z \xrightarrow{\wedge} e^{i\theta}} u(z)$ and $u^* \in L^p(T)$, $\forall p \in (0,1)$.

If the harmonic function $u : D \to R$ satisfies both conditions of Nikolski's theorem, then the analytic function $f(z) = u(z) + iv(z)$, where $u(z) = \operatorname{Re} f(z)$, will satisfy the conditions:

1) $\int_D |f(z)|\, d\sigma(z) < \infty$,

2) $\int_D |f'(z)|^2\, d\sigma(z) < +\infty$.

Indeed, by the Riesz inequality (see [1]) we have

$$\|f\|_1 = \int_D |f(z)|\, d\sigma(z) \le c\int_D |u(z)|\, d\sigma(z) = c\|u\|_1,$$

where c is a positive constant, whence we obtain

$$\|v\|_1 \le \|f\|_1 \le c\|u\|_1.$$

This means that the function v satisfies condition 1) of Nikolski's theorem.

Therefore, since

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = (grad\, u)^2,$$

the functions f and v satisfy condition 2) of Nikolski's theorem.

According to Theorem 6, there exist a.e. on the circumference T the angular limits:

$$f^*(t) = \lim_{z \xrightarrow{\wedge} t} f(z),$$

$$u^*(t) = \lim_{z \xrightarrow{\wedge} t} u(z),$$

$$v^*(t) = \lim_{z \xrightarrow{\wedge} t} v(z).$$

Moreover, $f^* \in L^p(T)$, $u^* \in L^p(T)$ and $v^* \in L^p(T)$, $\forall p \in (0,1)$. By Nikolski's theorem (see [5]), $f^* \in L^2(T)$, $u^* \in L^2(T)$ and $v^* \in L^2(T)$. Hence, by Smirnov's theorem (see [4]) $f \in H^2(D)$ and therefore $u, v \in h^2(D)$.

We have thereby proved the validity of

Theorem 7. *If the harmonic function* $u : D \to R$ *satisfies the conditions:*

1) $\int_D |u(z)| d\sigma(z) < \infty$,

2) $\int_D (grad\, u)^2 \, d\sigma(z) < +\infty$,

then $u \in h^2(D)$ *or*

$$\|u\| = \sup_{o \le r < 1} \int_0^{2\pi} \left| u\left(re^{i\theta}\right) \right|^2 d\theta < \infty .$$

Theorem 8. *If the real part of the analytic function* $f(z) = u(z) + iv(z)$ *satisfies the conditions:*

1) $\int_D |u(z)| d\sigma(z) < \infty$,

2) $\int_D (gradu)^2 \, d\sigma(z) < \infty$,

then $f \in H^2(D)$, *i.e.*

$$\sup_{0 \le r < 1} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta < \infty .$$

მათემატიკა

დირიხლეს სასრული ინტეგრალის მქონე ანალიზური და ჰარმონიული ფუნქციების შესახებ ერთეულოვან წრეში

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(წარმოდგენილია აკადემიის წევრის რ. ბანცურის მიერ)

სტატიაში შესწავლილია ერთეულოვან წრეში ჯამებადი წარმოებულის მქონე ანალიზური და ჯამებადი გრადიენტის მქონე ჰარმონიული ფუნქციების ინტეგრალური წარმოდგენები და სასაზღვრო თვისებები. დაზუსტებულია ჰარმონიული ფუნქციის სასაზღვრო მნიშვნელობის არსებობის შესახებ ს.მ. ნიკოლსკის ერთი თეორემა.

REFERENCES

1. *S. V. Shvedenko* (1985), Mathematical Analysis, **23**: 3-124, VINITI, M. (in Russian).
2. *V. Kabaila* (1970), Litovsk. Mat. Sb., **10**: 471-490 (in Russian).
3. *G. A. Oniani* (1978), Soobshch. AN GSSR, **89,** 2: 301-304 (in Russian).
4. *I. I. Privalov* (1950), Boundary properties of analytic functions, 2nd ed. M.-L. (in Russian).
5. *S. M. Nikolski* (1959), Boundary values of functions. In: USSR during 40 years: 1917–1957, Moscow (in Russian).

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