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Mathematics

Singular Integral Equation Arising from the Theory of the Penetration of Gamma Rays

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ABSTRACT. The aim of this paper is to study, in the class of Hölder functions, the linear integral equation arising from the theory of the penetration of gamma rays. Using the theory of singular integral equations, the solution of this equation is reduced to the Volterra equation. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *singular equation, piecewise holomorphic function, limiting values.*

We present a method for solving singular integral equations which frequently occur when investigating many important problems of mathematical physics, namely in the fundamental problems of nuclear transport theory [1-3]. This equation has the form

$$(Lu)(x,t) \equiv \int_a^x \int_{-1}^{+1} \frac{sK(x,y)}{s-t} u(y,s)\, ds\, dy + u(x,t) + \int_a^x \int_{-1}^{+1} \frac{tK(x.y)}{t-s} u(y,t)\, ds\, dy = f(x,t), \tag{1}$$

$$x \in [a,b],\ t \in (-1,+1),$$

where $K(x,y)$ is the real-valued continuous function, the right part $f(x,t)$ is the real-valued function satisfying H^* condition (Muskhelisvili class) [4] with respect to *t* and we look for the solution of a continuous function satisfying H^* condition with respect to *t*.

Let Ω_z be the integral operator defined by

$$(\Omega_z g_z)(x,t) \equiv g_z(x,t) + z \int_a^x \int_{-1}^{+1} \frac{K(x,y)}{z-t} g_z(y,t)\, dt\, dy, \quad x \in [a,b],\ t \in (-1,+1), \tag{2}$$

where the parameter *z* is any point on the plane. This operator operating on any continuous function $g_z(x,t)$

piecewise holomorphic in z with a cut on the real axis $[-1,+1]$ and satisfying the H^* condition in t, will define a piecewise holomorphic function with a cut on $[-1,+1]$. By using the Plemelj-Sokhotskii formulas, we can calculate the limiting values of Ω_z as

$$(\Omega^{\pm}{}_{\varsigma} g^{\pm}{}_{\varsigma})(x,t) \equiv g^{\pm}{}_{\varsigma}(x,t) + \varsigma \int_a^x \int_{-1}^{+1} \frac{K(x,y)}{\varsigma - s} g^{\pm}{}_{\varsigma}(y,s)\, ds\, dy \pm i\pi \int_a^x K(x,y) g^{\pm}_{\varsigma}(y,t) dy, \tag{3}$$

$$\varsigma \in (-1,+1), \quad x \in [a,b], \quad t \in (-1,+1).$$

The basic result for L is the following theorem.

Theorem 1. *The equation*

$$(Lu)(x,t) = f(x,t) \tag{4}$$

admits only a unique continuous solution $u(x,t)$ satisfying the H^* condition with respect to t.

Proof. Let u be a solution of (1) and consider the function

$$\Psi_z(x) = \frac{1}{2i\pi} \int_a^x \int_{-1}^{+1} \frac{sK(x,y)}{s-z} u(y,s)\, ds\, dy.$$

This function possesses the following properties:

(P_1) In the plane with the cut $[-1,+1]$ it is piecewise holomorphic with respect to z.

(P_2) As $z \to \infty$ it vanishes uniformly in x.

(P_3) By using the Plemelj-Sokhotskii formulas

$$\Psi_t^{\pm}(x) = \frac{1}{2i\pi} \int_a^x \int_{-1}^{+1} \frac{sK(x,y)}{s-t} u(y,t)\, dt\, dy \pm \frac{t}{2} \int_a^x K(x,y) u(y,t)\, dy.$$

Combining the above equalities with (3), we get

$$(\Omega^{+}{}_t \Psi^{+}{}_t)(x) - (\Omega_t^{=} \Psi_t^{-})(x) \equiv t \int_a^x K(x,y) u(y,t)\, dy + \int_a^x \int_{-1}^{+1} \frac{sK(x,y)}{s-t} \int_a^y K(y,y') u(y,s)\, dy'\, ds\, dy +$$

$$+ \int_a^x K(x,y) \int_a^y \int_{-1}^{+1} \frac{tK(y,y')}{s-t} u(y,s) ds dy' dy.$$

In view of (1) we obtain

$$(\Omega^{+}{}_t \Psi^{+}{}_t)(x) - (\Omega_t^{=} \Psi_t^{-})(x) = t \int_a^x K(x,y) f(y,t)\, dy.$$

Consequently, taking the Plemelj-Sokhotskii formulas into account, we can write

$$(\Omega_z \Psi_z)(x) = \int_a^x \int_{-1}^{+1} \frac{sK(x,y) f(y,t)}{s-z} ds\, dy. \tag{5}$$

By virtue of Tamarkin's theorem [5] it follows that there is a unique solution of (5). This solution possesses the properties:

(R_1) In the plane with a cut $[-1,+1]$ it is piecewise holomorphic with respect to z.

(R_2) As $z \to \infty$ it vanishes uniformly in x.

(R_3) It may be represented in the form

$$\Psi_z(x) = \frac{1}{2i\pi}\int_{-1}^{+1}\frac{k(x,t)}{s-z}ds\,, \quad z \notin [-1,+1]\,, \quad x \in [a,b]\,,$$

where $k(x,t)$ is a certain uniquely determined function.

In view of the Plemelj-Sokhotskii formulas, the following equality is valid for the limiting values:

$$\Psi_t^+(x) + \Psi_t^-(x) = \frac{1}{i\pi}\int_{-1}^{+1}\frac{\Psi_s^+(x) - \Psi_s^-(x)}{s-t}ds. \tag{6}$$

By using (3), from (6) we are able to write

$$\overline{\Psi}_t(x) + \int_a^x\int_{-1}^{+1}\frac{sK(x,y)}{s-t}\overline{\Psi}_t(y)\,ds\,dy + \int_a^x\int_{-1}^{+1}\frac{sK(x,y)}{s-t}\overline{\Psi}s(y)\,ds\,dy = \int_a^x K(x,y)f(y,t)\,dy\,, \tag{7}$$

where

$$\overline{\Psi}_t(x) = \Psi_t^+(x) - \Psi_t^-(x)\,.$$

Let us consider now the following equation

$$u(x,t) + \int_a^x\int_{-1}^{+1}\frac{tK(x,y)}{s-t}ds\; u(x,t)\,dy = \int_{-1}^{+1}\frac{s\overline{\Psi}_t(x)}{t-s}ds + f(x,t)\,, \tag{8}$$

which has a unique solution. Denote by

$$\mathrm{P}(x,t) = \overline{\Psi}_t(x) - \int_a^x K(x,y)u(y,t)\,dy\,.$$

In view of (7) from (8) it follows that

$$\mathrm{P}(x,t) + \int_a^x\int_{-1}^{+1}\frac{tK(x,y)}{s-t}\mathrm{P}(y,s)\,ds\,dy = 0\,.$$

But Ω_z has no eigenvalues on [-1,+1] and consequently

$$\overline{\Psi}_{t_0}(x) = \int_a^x K(x,y)u(y,t)\,dy\,.$$

Taking into account this last assertion, from (8) we obtain

$$u(x,t) + \int_a^x\int_{-1}^{+1}\frac{tK(x,y)}{t-s}ds\; u(x,t)\,dy = \int_a^x\int_{-1}^{+1}\frac{sK(x,y)}{t-s}u(y,s)\,ds\,dy + f(x,t)\,,$$

which means that (4) holds and the proof is complete.

Now, our aim is a deeper study of the singular operator L. We wish to find the reduction operator of L and hence reduction of the considered equation to a regular equation. For this purpose, we need to introduce the following singular operator

$$(Su)(x,t) \equiv \int_a^x \int_{-1}^{+1} \frac{sK(x,y)}{t-s} u(y,s)\,ds\,dy + u(x,t) + \int_a^x \int_{-1}^{+1} \frac{tK(x,y.)}{t-s} u(y,t)\,ds\,dy \,.$$

We shall note the following property of the introduced operator. For any two continuous functions $u(x,t)$ and $v(x,t)$, satisfying the H^* conditions with respect to t

$$\int_a^b \int_{-1}^{+1} uSv\,dt\,dx = \int_a^b \int_{-1}^{+1} vLu\,dt\,dx \,.$$

Consequently, if equation (1) has a solution, then necessarily

$$\int_a^b \int_{-1}^{+1} vf\,dt\,dx = 0 \,,$$

where v is any solution of the homogeneous equation

$$Sv = 0 \,. \tag{9}$$

But the equation (1) admits only a unique solution, therefore equation (9) also admits a unique zero solution.

In our further consideration we will need also the following identity:

$$\int_a^x \int_{-1}^{+1} \frac{sK(x,y)}{t-s} \int_a^y \int_{-1}^{+1} \frac{s'K(y,y')}{s'-s} u(y',s')\,ds'\,dy'\,ds\,dy =$$

$$= \int_a^x \int_{-1}^{+1} \frac{u(y',t')}{s'-t} \left(\int_y^x \int_{-1}^{+1} \frac{tK(x,y)s'K(y,y')}{t-s} ds'dy' - \int_y^x \int_{-1}^{+1} \frac{tK(x,y)s'K(y,y')}{s'-s} ds'dy' \right) dsdy +$$

$$+\pi^2 t^2 \int_a^x \int_y^x K(x,y)K(y,y')u(y',t)dy'dy \,. \tag{10}$$

This identity can be obtained by using the Bertrand-Poincare formula [4].

We can show the important property of the operator S in the Theorem below.

Denote

$$R(x,y,t) = -\pi^2 t^2 \int_y^x K(x,y')K(y',x)dy' + \int_{-1}^{+1} \frac{tK(x,y)}{t-s} ds + \int_{-1}^{+1} \frac{tK(y,x)}{t-s} ds -$$

$$-\int_y^x \int_{-1}^{+1} \frac{tK(x,y)}{t-s} ds \int_{-1}^{+1} \frac{tK(y',x)}{t-s} dsdy' \,.$$

Theorem 2. *The composition SL contains no singular part and the equality*

$$(SL)(x,t) = u(x,t) + \int_a^x R(x,y,t)u(y,t)\,dy \tag{11}$$

holds.

Proof. Performing the operations indicated on the left-hand side of (11) using identity (10) we obtain that Eq. (11) is true.

Thus, we may say that S reduces operator of L.

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