Mathematics

The Nonlinear Characteristic Goursat Problem for a Nonlinear Oscillation Equation

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ABSTRACT. The paper proposes a nonlinear analogue of the characteristic Goursat problem for a nonlinear oscillation equation, which makes it possible to simultaneously define regular solutions and extension domains. The structures of these domains up to the proximity of the points of degeneracy of degree of the equation are described. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *characteristic invariants, general integral, solution definition domain.*

In the works [1, 2] dedicated to linear non-strictly hyperbolic equations much attention is given to the well-known second order equation of nonlinear oscillations

$$u_y^4 u_{xx} - u_{yy} = 0,$$

which depending on the behavior and values of the first order derivative u_y of the sought solution $u(x, y)$ may generate parabolically. Its general integral is represented by means of the generator of groups of solutions which are based on contact transformations. However the question as to the solvability of problems was not considered. If the method proposed there is used for construction of a general integral for an equation with the non-zero right-hand part, then we are faced with great and even insurmountable difficulties.

In the present paper, an analogous question is considered for the equation

$$u_y^4 u_{xx} - u_{yy} = cx^{-2} u_y^4, \qquad c = const \tag{1}$$

which is interesting not only from the theoretical viewpoint, but also as having various practical applications [3-10].

Like in the case for general nonlinear hyperbolic equations, for equation (1) too, linear formulations of boundary value or characteristic problems, except for the Cauchy initial boundary value problem, are meaningless. This fact is caused by the dependence of characteristic families on yet unknown solutions.

The right-hand part of equation (1) along the ordinate axis is unbounded. This property makes it possible to attribute this equation to the class of Euler-Darboux equations [11-12]. By performing multiplication by a factor defining this unboundedness and making some assumptions, we can consider, instead of (1), an equation with the degeneration not only of hyperbolicity but of order, too. However, there may exist solutions along which equation (1) remains hyperbolic. In other words, these are solutions for which the characteristic directions defined by the characteristic roots

$$\lambda_1 = u_y^{-2}, \quad \lambda_2 = -u_y^{-2},$$

do not coincide anywhere. Naturally, the class of hyperbolic solutions of equation (1) is defined by the condition

$$0 \neq \left| u_y(x, y) \right| < \infty .$$

Before formulating a nonlinear analogue of the Goursat problem, let us clarify the general properties of hyperbolic solutions of equation (1). For this we will use the classical characteristic method [13].

As is known, the characteristic roots λ_1, λ_2 give the differential relations of characteristic directions

$$u_y^2 dy - dx = 0, \quad u_y^2 dy + dx = 0 . \tag{2}$$

If equation (1) is considered with (2) taken into account, then we come to the following differential characteristic relations

$$x^2 u_y^4 \, du_x - x^2 u_y^2 \, du_y - c u_y^4 \, u \, dx = 0, \quad x^2 u_y^4 \, du_x + x^2 u_y^2 \, du_y - c u_y^4 \, u \, dx = 0 .$$

The following theorem is valid.

Theorem 1. *Assuming that* $c > -\frac{1}{4}$, *each of the characteristic systems of equation* (1) *admits exactly two first integrals and they are defined in the explicit form*

$$\begin{cases} \xi \equiv \left(u_y^{-1} + u_x\right) x^{\alpha} - \alpha \, u \, x^{\alpha-1} \\ \xi_1 \equiv \left(u_y^{-1} + u_x\right) x^{1-\alpha} - (1-\alpha) u \, x^{-\alpha} \end{cases} \tag{3}$$

for the family of the root λ_1 *and*

$$\begin{cases} \eta \equiv \left(u_y^{-1} - u_x\right) x^{\alpha} + \alpha \, u \, x^{\alpha-1} \\ \eta_1 \equiv \left(u_y^{-1} - u_x\right) x^{1-\alpha} + (1-\alpha) u \, x^{-\alpha} \end{cases}, \quad \alpha = \frac{1}{2}\left(1 + \sqrt{4c+1}\right) \tag{4}$$

for the family of the root λ_2 .

Due to these two pairs of first integrals (ξ, ξ_1) and (η, η_1) which are actually characteristic invariants, it follows that in the class of hyperbolic solutions we can construct two intermediate integrals

$$\xi_1 = \varphi'(\xi), \quad \eta_1 = \psi'(\eta)$$

of equation (1) [13]. In these integrals we denote by φ, ψ arbitrary functions smooth enough to ensure the differentiability of the sought solution up to second order.

Theorem 2. *If* $\varphi,\ \psi \in C^3(R_1)$, *then equation* (1) *is equivalent to a triple of the following relations*

$$x=\left(\frac{\varphi'(\xi)+\psi'(\eta)}{\xi+\eta}\right)^{\frac{1}{1-2\alpha}}, \tag{5}$$

$$y=\frac{1}{4(1-2\alpha)}\left[(\xi+\eta)(\psi'(\eta)-\varphi'(\xi))+2(\varphi(\xi)-\psi(\eta))\right], \tag{6}$$

$$u=\frac{1}{1-2\alpha}\left[\xi\left(\frac{\varphi'(\xi)+\psi'(\eta)}{\xi+\eta}\right)^{\frac{1-\alpha}{1-2\alpha}}-\varphi'(\xi)\left(\frac{\varphi'(\xi)+\psi'(\eta)}{\xi+\eta}\right)^{\frac{\alpha}{1-2\alpha}}\right], \tag{7}$$

which in a certain sense can be taken as a general integral of equation (1)*, whereas the invariants* ξ, η *play the role of characteristic variables.*

Thus, the invariants ξ, ξ_1 will be constant along each characteristic of the family of the root λ_1. The invariants η, η_1 will preserve constant values along the characteristics of the family of the root λ_2. This circumstance should be taken into consideration when formulating the problems with regard to the question whether characteristics are included in the data carrier. Based on this principle, we propose the following nonlinear variant of the Goursat problem.

Suppose we are given two arcs γ_1, γ_2 drawn from the common point $(a, f(a))$ and let them be given in the explicit form

$$\gamma_1 :\ y=f_1(x),\quad a\le x\le b,\quad a<0,\quad b>0$$

and

$$\gamma_2 :\ y=f_2(x),\quad a\le x\le d,\ \ d>0,\quad f_1(a)=f_2(a). \tag{8}$$

Assume that the functions f_1 and f_2 are three times continuously differentiable and the arc γ_1 monotonically ascends, whereas the arc γ_2, vice versa, monotonically descends.

The Goursat problem consists in constructing the solution of a hyperbolic equation using data whose carriers consist only of characteristics. In that case, the problem conditions will be the assumptions that the arc γ_1 is a characteristic of the family of the root λ_1, whereas the arc γ_2 is a characteristic of the family of the root λ_2. But since the characteristic directions of these families at every point are symmetrical with respect the abscissa axis direction, it immediately follows that the requirement

$$f_1'(a)=-f_2'(a)$$

is justified.

The Goursat Problem. Find a regular hyperbolic solution $u(x,y)$ of equation (1) and, simultaneously with it, a domain of its extension when the curves γ_1 and γ_2 are the arcs of the characteristics, and the values

$$u(a, f_1(a))=\vartheta, \tag{9}$$

$$u_x\left(a, f_1(a)\right) = \delta \tag{10}$$

are given at the common point.

The assumption that the arcs γ_1 and γ_2 are drawn from the point $(a,0)$ does not cause the loss of generality. Therefore we assume

$$f_1(a) = f_2(a) = 0.$$

By the conditions of the problem we have

$$f_1'(x) = u_y^{-2}, \quad f_2'(x) = -u_y^{-2},$$

which define the values of the derivative u_y on γ_1 and γ_2. But they are defined non-uniquely:

$$u_y\Big|_{\gamma_1} = \pm\frac{1}{\sqrt{f_1'(x)}}, \quad u_y\Big|_{\gamma_2} = \pm\frac{1}{\sqrt{-f_2'(x)}}.$$

To obtain the continuity of the derivative, we should take the right-hand parts of the latter equalities of the same sign.

Let us first consider the case

$$u_y\left(a, f_1(a)\right) = \frac{1}{\sqrt{f_1'(a)}}. \tag{11}$$

Using (9)-(11) we calculate in a straightforward manner the values of all characteristic invariants at the point $(a,0)$ and introduce for them the notation

$$\xi\Big|_{(a,0)} \equiv \xi^{[a]}, \quad \xi_1\Big|_{(a,0)} \equiv \xi_1^{[a]}, \quad \eta\Big|_{(a,0)} \equiv \eta^{[a]}, \quad \eta_1\Big|_{(a,0)} \equiv \eta_1^{[a]}.$$

By the property of characteristic invariants that they are constant along the respective curves, we have

$$\xi\Big|_{\gamma_1} \equiv \xi^{[a]}, \quad \xi_1\Big|_{\gamma_1} \equiv \xi_1^{[a]} \tag{12}$$

and

$$\eta\Big|_{\gamma_2} \equiv \eta^{[a]}, \quad \eta_1\Big|_{\gamma_2} \equiv \eta_1^{[a]}. \tag{13}$$

Let us now turn to the general integral (5)-(7) defined in terms of the characteristic invariants ξ, η and, using (5), (7), establish the relation between the values *x*, *u* along the arc γ_1:

$$u\Big|_{\gamma_1} = \frac{1}{1-2\alpha}\left[\xi^{[a]}x^{1-\alpha} - \xi_1^{[a]}x^{\alpha}\right]. \tag{14}$$

From the second relation (12) we obtain

$$u_x\Big|_{\gamma_1} = \frac{1-\alpha}{1-2\alpha}\xi^{[a]}x^{-\alpha} - \frac{\alpha}{1-2\alpha}\xi_1^{[a]}x^{\alpha-1} - \sqrt{f_1'(x)}. \tag{15}$$

Substituting the derivative value

$$u_y\Big|_{\gamma_2} = \frac{1}{\sqrt{-f_2'(x)}}$$

into the first equality (13), we express the relation between the values x, u, u_x on γ_2 as follows

$$u_x\Big|_{\gamma_2} = -\eta^{[a]}x^{-\alpha} + \alpha\, u\, x^{-1} + \sqrt{-f_2'(x)}\,. \tag{16}$$

Substituting the values u_x, u_y into the total differential

$$du = \left(u_x + f_2'(x)u_y\right)dx$$

taken along γ_2, we obtain

$$\frac{du}{dx}\Big|_{\gamma_2} = \frac{\alpha}{x}u\Big|_{\gamma_2} - \eta^{[a]}x^{-\alpha}, \quad x \in [a,b]. \tag{17}$$

Let us consider relation (17) as an ordinary differential equation with respect to the trace $U(x) = u(x, f_2(x))$ of the function u on γ_2

$$\frac{dU}{dx} - \frac{\alpha}{x}U = -\eta^{[a]}x^{-\alpha}$$

and write explicitly the solution of the Cauchy problem with the initial condition $U(a) = \vartheta$. We obtain

$$u\Big|_{\gamma_2} = \left(\vartheta a^{-\alpha} + \frac{a^{1-2\alpha}}{1-2\alpha}\eta^{[a]}\right)x^{\alpha} - \eta^{[a]}\frac{x^{1-\alpha}}{1-2\alpha}. \tag{18}$$

A pair of relations (13), (18) enables us to define the relation of the derivative u_x with the variable x along the arc γ_2 as follows

$$u_x\Big|_{\gamma_2} = \frac{\alpha-1}{1-2\alpha}\eta^{[a]}x^{-\alpha} + \alpha\left(\vartheta a^{-\alpha} + \frac{a^{1-2\alpha}}{1-2\alpha}\eta^{[a]}\right)x^{\alpha-1} + \sqrt{-f_2'(x)}\,. \tag{19}$$

Thus the conditions of problem (1), (9), (10) make it possible to define along both characteristic arcs γ_1, γ_2 the interdependence between the argument x, the solution $u(x,y)$ of equation (1) and its first order derivatives. But the aim we pursue is to define the solution not only on the characteristics but also outside the data carrier. To this end, we will again try to use the general properties of characteristic invariants. On the arc γ_1, we can represent the invariants η, η_1 from the other family as functions of the argument x. This is done by substitution of (11), (14), (15) into (4). On the arc γ_2, the values ξ, ξ_1 are constructed in the same manner.

From an arbitrary point $P_2(x_2, f(x_2)) \in \gamma_2$ let us draw the characteristic γ_3 of the family of the root λ_1. The constants along will be the invariants ξ and ξ_1. Another point $P_1(x_1, f_1(x_1)) \in \gamma_1$ is treated analogously and the characteristic of the family of the root λ_2 drawn from it is denoted by γ_4. The values of the invariants η, η_1 at the point P_1 are preserved all along the arc γ_4. According to the general theory, a set of points of intersection of the analogous characteristics γ_3, γ_4 defines the domain of solution extension. If there exits a point of intersection of the arcs γ_3, γ_4, we denote it by $M\left(x^0, y^0\right)$.

Let us introduce the notation

$$\xi|_{P_2} \equiv \xi^{[x_2]},\quad \xi_1|_{P_2} \equiv \xi_1^{[x_2]},\quad \eta|_{P_1} \equiv \eta^{[x_1]},\quad \eta_1|_{P_1} \equiv \eta_1^{[x_1]}.$$

As has already been said, the following relations hold true

$$\xi|_{\gamma_3} = \xi^{[x_2]},\quad \xi_1|_{\gamma_3} = \xi_1^{[x_2]}, \tag{20}$$

$$\eta|_{\gamma_4} = \eta^{[x_1]},\quad \eta_1|_{\gamma_4} = \eta_1^{[x_1]} \tag{21}$$

and they are simultaneously fulfilled at the point $M\left(x^0, y^0\right)$ of intersection of the arcs γ_3, γ_4.

Thus we obtain the following system with respect to x^0, $u^0 = u\left(x^0, y^0\right)$, $u_x^0 = u_x\left(x^0, y^0\right)$, $u_y^0 = u_y\left(x^0, y^0\right)$:

$$\begin{cases} \left(u_y^{0^{-1}} + u_x^0\right) x^{0\alpha} - \alpha u^0 x^{0\alpha-1} = \xi^{[x_2]} \\ \left(u_y^{0^{-1}} + u_x^0\right) x^{01-\alpha} - (1-\alpha) u^0 x^{0-\alpha} = \xi_1^{[x_2]} \\ \left(u_y^{0^{-1}} - u_x^0\right) x^{0\alpha} + \alpha u^0 x^{0\alpha-1} = \eta^{[x_1]} \\ \left(u_y^{0^{-1}} - u_x^0\right) x^{01-\alpha} - (1-\alpha) u^0 x^{0-\alpha} = \eta_1^{[x_1]} \end{cases}. \tag{22}$$

We have already dealt with a system of form (22) when constructing the general integral (7)-(8). Hence to solve system (22), we perform analogous actions with the only difference that (22) will be considered not with respect to the functions x, u, u_x, u_y, and ux but with respect to their concrete values at the point $M\left(x^0, y^0\right)$. The right-hand parts are also concrete constant values well defined by the conditions of problem (1), (9), (10).

To find $y^0 = y\left(x_1, x_2\right)$ we resort to relation (6). Taking into consideration the explicit equations of the arcs γ_1, γ_2, we finally define the value $y = y^0$ in terms of x_1, x_2. The values x^0, y^0, u^0 are expressed as follows

$$x^0 = \left(\frac{\xi_1^{[x_2]} + \eta_1^{[x_1]}}{\xi^{[x_2]} + \eta^{[x_1]}}\right)^{\frac{1}{1-2\alpha}}, \tag{23}$$

$$y^0 = f_1(x_1) + f_2(x_2) +$$

$$+\frac{1}{4(1-2\alpha)}\left[\xi^{[x_2]}\eta_1^{[x_1]} - \eta^{[x_1]}\xi_1^{[x_2]} - \xi^{[a]}\eta_1^{[x_1]} + \eta^{[x_1]}\xi_1^{[a]} - \xi^{[x_2]}\eta_1^{[a]} + \xi_1^{[x_2]}\eta^{[a]} + \xi^{[a]}\eta_1^{[a]} - \eta^{[a]}\xi_1^{[a]}\right], \tag{24}$$

$$u^0 = \frac{1}{1-2\alpha}\left[\xi^{[x_2]}\left(\frac{\xi^{[x_2]}+\eta_1^{[x_1]}}{\xi^{[x_2]}+\eta^{[x_1]}}\right)^{\frac{1-\alpha}{1-2\alpha}} - \xi_1^{[x_2]}\left(\frac{\xi_1^{[x_2]}+\eta_1^{[x_1]}}{\xi^{[x_2]}+\eta^{[x_1]}}\right)^{\frac{\alpha}{1-2\alpha}}\right]. \tag{25}$$

We have obtained the definition domain *D* of the solution $u\left(x^0, y^0\right)$ of problem (1), (9), (10) for the current x_1, x_2. This domain is well defined by relations (23), (24), where the expressions of x^0, y^0 are presented depending on x_1, x_2. We consider the values of these functions as the current coordinates describing the domain *D*. Then, taking into consideration the expressions of invariants along the arcs γ_3, γ_4 and their values at $(a,0)$, we can represent the solution of the considered problem (1), (9), (10) by the following three equalities

$$x = F\left(x_1, x_2\right) \equiv \left(\frac{\sqrt{f_1'(x_1)}\,x_1^{1-\alpha} + \sqrt{-f_2'(x_2)}\,x_2^{1-\alpha} - \sqrt{f_1'(a)}\,a^{1-\alpha}}{\sqrt{f_1'(x_1)}\,x_1^{\alpha} + \sqrt{-f_2'(x_2)}\,x_2^{\alpha} - \sqrt{f_1'(a)}\,a^{\alpha}}\right)^{\frac{1}{1-2\alpha}}, \tag{26}$$

$$y = G\left(x_1, x_2\right) \equiv f_1\left(x_1\right) + f_2\left(x_2\right) + \frac{1}{1-2\alpha}\left[\left(\sqrt{f_1'(x_1)}\,x_1^{1-\alpha} - \sqrt{f_1'(a)}\,a^{1-\alpha}\right)\left(\sqrt{-f_2'(x_2)}x_2^{\alpha} - \sqrt{f_1'(a)}\,a^{\alpha}\right) + \right.$$

$$\left. + \left(\sqrt{f_1'(a)}\,a^{\alpha} - \sqrt{f_1'(x_1)}\,x_1^{\alpha}\right)\left(\sqrt{-f_2'(x_2)}x_2^{1-\alpha} - \sqrt{f_1'(a)}\,a^{1-\alpha}\right)\right], \tag{27}$$

$$u = \frac{1}{1-2\alpha}\left[\left(\delta a^{\alpha} - \alpha\vartheta a^{\alpha-1} + 2\sqrt{-f_2'(x_2)}\,x_2^{\alpha} - \sqrt{f_1'(a)}\,a^{\alpha}\right)\right] \times$$

$$\times\left[\left(\frac{\sqrt{f_1'(x_1)}\,x_1^{1-\alpha} + \sqrt{-f_2'(x_2)}\,x_2^{1-\alpha} - \sqrt{f_1'(a)}\,a^{1-\alpha}}{\sqrt{f_1'(x_1)}\,x_1^{\alpha} + \sqrt{-f_2'(x_2)}\,x_2^{\alpha} - \sqrt{f_1'(a)}\,a^{\alpha}}\right)^{\frac{1-\alpha}{1-2\alpha}} + \right.$$

$$+\left((\alpha-1)\vartheta a^{-\alpha} - \sqrt{f_1'(a)}\,a^{1-\alpha} + \delta a^{1-\alpha} + 2\sqrt{-f_2'(x_2)}\,x_2^{1-\alpha}\right) \times$$

$$\left. \times\left(\frac{\sqrt{f_1'(x_1)}\,x_1^{1-\alpha} + \sqrt{-f_2'(x_2)}\,x_2^{1-\alpha} - \sqrt{f_1'(a)}\,a^{1-\alpha}}{\sqrt{f_1'(x_1)}\,x_1^{\alpha} + \sqrt{-f_2'(x_2)}\,x_2^{\alpha} - \sqrt{f_1'(a)}\,a^{\alpha}}\right)^{\frac{\alpha}{1-2\alpha}}\right]. \tag{28}$$

They simultaneously define both the domain *D* and the sought solution.

Note that the characteristic of the family λ_1 lying in the domain $D \cap \left\{F\left(x_1, x_2\right) \le \varepsilon < 0\right\}$ and drawn from the fixed point $\left(x_2^0, f_2\left(x_2^0\right)\right)$ is parametrically represented by the equations

$$x = F\left(x_1, x_2^0\right), \quad y = G\left(x_1, x_2^0\right), \quad x_1 \in [a, \varepsilon].$$

The parametric equations of the characteristic of the family λ_2 drawn from the point $\left(x_1^0, f_1\left(x_1^0\right)\right)$ are given as follows

$$x = F\left(x_1^0, x_2\right), \quad y = G\left(x_1^0, x_2\right), \quad x_2 \in \left[a, \varepsilon\right].$$

For the characteristics of one and the same family not to intersect each other in the domain $D \cap \{F(x_1, x_2) \le \varepsilon < 0\}$, it is sufficient that the conditions

$$\left[F\left(x_1^0, x_2\right) - F\left(x_1', x_2\right)\right]^2 + \left[G\left(x_1^0, x_2\right) - G\left(x_1', x_2\right)\right]^2 \ne 0, \tag{29}$$

$$\left[F\left(x_1, x_2^0\right) - F\left(x_1, x_2'\right)\right]^2 + \left[G\left(x_1, x_2^0\right) - G\left(x_1, x_2'\right)\right]^2 \ne 0 \tag{30}$$

be fulfilled for any fixed values $x_1^0 \ne x_1'$ and $x_2^0 \ne x_2'$ from the interval $\left[a, \varepsilon\right]$ and for the parameters $x_1, x_2 \in \left[a, \varepsilon\right]$.

If

$$u_y\Big|_{\gamma_1} = -\frac{1}{\sqrt{f_1'(x)}}, \quad u_y\Big|_{\gamma_2} = -\frac{1}{\sqrt{-f_2'(x)}},$$

then by a reasoning analogous to that in the preceding case we see that problem (1), (9), (10) has, in addition to (26)-(28), one more solution represented by the formulas

$$\begin{aligned}
&x = F\left(x_1, x_2\right),\\
&y = G\left(x_1, x_2\right),\\
&u = \frac{1}{1-2\alpha}\Bigg[\left(\delta a^{\alpha} - \alpha \vartheta a^{\alpha-1} - 2\sqrt{-f_2'\left(x_2\right)}\, x_2^{\alpha} + \sqrt{f_1'\left(a\right)}\, a^{\alpha}\right) F^{\frac{1-\alpha}{1-2\alpha}}\left(x_1, x_2\right) +\\
&+\left(\left(\alpha-1\right)\vartheta a^{-\alpha} + \sqrt{f_1'\left(a\right)}\, a^{1-\alpha} + \delta a^{1-\alpha} - 2\sqrt{-f_2'\left(x_2\right)}\, x_2^{1-\alpha}\right) F^{\frac{\alpha}{1-2\alpha}}\left(x_1, x_2\right)\Bigg].
\end{aligned} \tag{31}$$

The definition domains of these solutions coincide since in both cases the arguments *x* and *y* are defined in the same manner.

The following theorem is true.

Theorem 3. *If conditions* (29), (30) *are fulfilled, then for any arbitrarily real branch of the multi-valued function* (26) *in the domain* $D \cap \left\{F\left(x_1, x_2\right) \le \varepsilon < 0\right\}$ *there exist regular hyperbolic solutions of problem* (1), (9), (10) *representable by formulas* (26)-(28) *and* (26), (27), (31).

მათემატიკა

გურსას არაწრფივი მახასიათებელი ამოცანა არაწრფივი რხევების განტოლებისათვის

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*** საქართველოს ტექნიკური უნივერსიტეტის ნიკო მუსხელიშვილის სახელობის გამოთვლითი მათემატიკის ინსტიტუტი, თბილისი*

(წარმოდგენილია აკადემიკოს ნ. ვახანიას მიერ)

ნაშრომში არაწრფივი რხევების ცნობილი განტოლებისათვის შესწავლილია გურსას მახასიათებელი ამოცანის ერთი არაწრფივი ვარიანტი, რომლის მიხედვითაც ერთდროულად უნდა განისაზღვროს რეგულარული ჰიპერბოლური ამონახსნი და მისი გავრცელების არე. აღწერილია ამ არეთა სტრუქტურული თვისებები განტოლების რიგის გადაგვარების წერტილთა სიახლოვემდე.

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