საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 6, №3, 2012
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 6, no. 3, 2012

Mathematics

Affine Isomorphisms of Power Groups

Tengiz Bokelavadze* and Alexander Lashkhi**

* *Faculty of Exact and Natural Sciences, Akaki Tsereteli State University, Kutaisi*
** *Department of Theoretical Informatics and Computer Modelling, Georgian Technical University,*

(Presented by Academy Member Hvedri Inassaridze)

ABSTRACT. In the present paper the lattice of cosets $\Re(G)$ is constructed for Hall's power group *G* over the ring *W*. This lattice is called the affine or coset lattice of *G*. Since in the lattice $\Re(G)$ only the elements of G cover $\varnothing$, the isomorphism $f:\Re(G)\to\Re(G_1)$ defines the bijection $f:G\to G_1$. Among all possible isomorphisms *f* we shall select in the sequel those for which $f(1)=1$. Such isomorphisms will be called the affine isomorphisms. We prove the following theorem: Let $f:\Re(G)\to\Re(G_1)$ be an affine isomorphism between the *w*-power groups *G* and G_1 over the fields *W* and W_1, respectively, if *G* is nilpotent of class 2 then *f* is either a semilinear isomorphism or a semilinear antiisomorphism with respect to the isomorphism $\sigma:W\to W_1$. The given example shows that the theorem is false for the class of nilpotency ≥ 3. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *affine geometry, coset lattice, power groups.*

Let *L* be a lattice and *M* a subset of *L* which is a lattice relative to the induced partial order. Clearly, *M* is not a sublattice of *L*; the subgroup lattice of a group *G* as a subset of the lattice of all subsets of *G* is an example. If $\cap$ and $\wedge$ denote the intersections in *L* and *M*, respectively, then for $x,y\in M$, $x\wedge y$ is a lower bound of $\{x,y\}$ in *L* and so $x\wedge y\leq x\cap y$. We call *M* a *meet-sublattice* of *L* if $x\wedge y\leq x\cap y$ for all $x,y\in M$. The lattice *M* is a *complete meet-sublattice* of *L* if *M* and *L* are complete lattices and for every subset $S\subset M$ the greatest lower bounds of *S* in *M* and *L* coincide.

Proposition. *If M is a subset of a complete lattice L such that for every subset S of M the greatest lower bound $\cap S$ of S in L is contained in M, then M is a complete meet-sublattice of L.*

Proof. For a subset *S* of *M* let S^* be the set of upper bounds of *S* in *M*. By hypothesis, $\inf S^*\in M$. Every $s\in S$ is a lower bound of S^*, hence $s\leq\inf S^*$. Thus $\inf S^*$ is an upper bound of *S* and then, certainly, the least upper bound of *S* in *M*. Clearly, inf *S* is the greatest lower bound of *S* in *M*. Hence *M* is a complete lattice and the greatest lower bounds of S in M and L coincide.

If G is a group, the subgroup lattice $L(G)$ is an example of a complete meet-sublattice of the lattice of all subsets of G. We introduce two further lattices of this kind.

The notion of a discrete w-power group was introduced by Hall [1]. At the present time there are a number of works where the properties of w-power groups are studied. In [2] the fundamental theorem of projective geometry for w-power groups was proved for principal ideal domains which are not fields. For the relation problems in Lie algebras and groups see [3-6]. For affine geometry of modules and the fundamental theorem see [7]. In [8,9] the affine lattices and the fundamental theorem of affine geometry for Lie algebras and power groups are studied. Furthermore, G (and G_1) is Hall's power group over the ring W (the ring W_1).

Let $\Re(G)$ be the set of all right cosets of subgroups of G together with the empty set $\varnothing$. If $\mathcal{Y}$ is a nonempty subset of $\Re(G)$, then either $\cap\mathcal{Y}=\varnothing\in\Re(G)$ or there exists an element $x\in\cap\mathcal{Y}$. In the latter case, $\mathcal{Y}=\{H_i x \mid i\in I\}$ for certain subgroups H_i of G, and hence $\cap\mathcal{Y}=\left(\bigcap_{i\in I} H_i\right)x\in\Re(G)$. If $\mathcal{Y}$ is an empty subset of $\Re(G)$, then $\cap\mathcal{Y}=G\in\Re(G)$. The lattice $\Re(G)$ is a complete meet-sublattice of the lattice of all subsets of G. Since $Hx=x\left(x^{-1}Hx\right)$ for all $H\le G$ and $x\in G$, every right coset is a left coset, and conversely. So $\Re(G)$ is the set of all (right or left) cosets in G and we call $\Re(G)$ the *affine or coset lattice* of G. The subgroups of G are precisely the cosets containing the coset 1. Hence $L(G)$ is the interval $[G,1]$ in $\Re(G)$.

For the basic properties of affine lattice $\Re(G)$ see [4].

Affine lattices do not usually have nice lattice properties. For example, if $1<H<G$ and $a\in G/H$, then $\{\varnothing,\{a\},Ha,\langle H,a\rangle,H\}$ is a nonmodular sublattice (pentagon) of $\Re(G)$. Thus a group with a modular coset lattice is cyclic without proper subgroups; in this connection see [2-4, 6, 8, 9].

Thus the groups whose coset lattices have special properties, is not a very fruitful one: almost any interesting lattice property is satisfied by nearly all or by a very few coset lattices, or else it holds in $\Re(G)$ if and only if it holds in $L(G)$ [2, 4-6, 8, 9]. The study of isomorphisms between coset lattices is more interesting. Since the atoms of $\Re(G)$ are the one-element subsets of G and for any such atom $\{g\}$, the interval $[G,\{g\}]$ in $\Re(G)$ is isomorphic to $L(G)$, two groups with isomorphic coset lattices have the same order and isomorphic subgroup lattices. Nevertheless there are nonisomorphic groups with isomorphic coset lattices. An isomorphism $\sigma:\Re(G)\to\Re(\bar{G})$ can be regarded as a bijective map from G to $\bar{G}$ preserving cosets. We consider under which condition σ is a semilinear isomorphism from G to $\bar{G}$.

Let G and G_1 be w-power groups over the rings W and W_1, respectively. The bijection $f:G\to G_1$ will be called a semilinear isomorphism with respect to the isomorphism $\sigma:W\to W_1$ if the equality

$$f\left(x_1^{\alpha_1}x_2^{\alpha_2}\right)=f\left(x_1\right)^{\sigma(\alpha_1)}f\left(x_2\right)^{\sigma(\alpha_2)}$$

is fulfilled for any $x_1,x_2\in G$ and $\alpha_1,\alpha_2\in W$ and f will be called a semilinear antiisomorphism if the equality

$$f\left(x_1^{\alpha_1}x_2^{\alpha_2}\right)=f\left(x_2\right)^{\sigma(\alpha_2)}f\left(x_1\right)^{\sigma(\alpha_1)}$$

is fulfilled for any $x_1,x_2\in G$ and $\alpha_1,\alpha_2\in W$.

We say that the fundamental theorem of projective geometry is valid for *w*-power group *G* over the ring *W*, if the lattice isomorphism $\varphi:L(G)\to L(G_1)$ (where $L(G)$ and $L(G_1)$ are the lattices of all *w*-subgroups of *G* and G_1) implies the existence of a semilinear isomorphism $f:G\to G_1$ with respect to the isomorphism $\alpha:W\to W_1$ such that $f(A)=\varphi(A)$ for all $A\in L(G)$.

Since in the lattice $\Re(G)$ only the elements of *G* cover $\varnothing$, the isomorphism $f:\Re(G)\to\Re(G_1)$ defines the bijection $f:G\to G_1$.

Among all possible isomorphisms *f* we shall select in the sequel those for which $f(1)=1$. Such isomorphisms will be called affine isomorphisms. If $f:\Re(G)\to\Re(G_1)$ is an isomorphism, then the bijection φ defined by the equality

$$\varphi(x)=f(x)\left[f(1)\right]^{-1}$$

will be an affine isomorphism.

We say that the fundamental theorem of affine geometry is valid for a *w*-power group *G* if any affine isomorphism is either a semilinear isomorphism or a semilinear antiisomorphism.

Remark 1. If $a\in A$ is a fixed element and $f(a)=a_1$ then the mapping

$$\varphi(x)=f(ax)\left[f(1_a)\right]^{-1}$$

is an affine isomorphism. Indeed, φ will be a *C*-isomorphism defined by the element a_1^{-1} i.e. it will be an automorphism $\left[\tilde{a}_1\right]^{-1}\in Aut\left[\Re(G)\right]$; since $\varphi(1)=f(a)a_1^{-1}$, we have that f_1 is an affine isomorphism.

Not each affine isomorphism (antiisomorphisms) is a semilinear isomorphism. Any one-dimensional vector space over Z_p admits $(p-1)!$ affine automorphisms while the group of all internal automorphisms of Z_p has order $p-1$. Therefore for $p>3$ a one-dimensional vector space over Z_p admits affine automorphisms different from ordinary ones.

Lemma 1. *Let* $\Re(G)\to\Re(G_1)$ *be an affine isomorphism. Then the following statements are true:*

(i) *f induces a lattice isomorphism* $f:L(G)\to L(G_1)$;

(ii) $f(\langle M\rangle)=\langle f(M)\rangle$ *for any subset* $M\subseteq G$;

(iii) $f(a\langle b\rangle)=f(a)\langle f(b)\rangle$ *for any* $a,b\in G$.

Assume that *W* is a commutative domain. The *w*-group *G* is called torsion-free if $x^\alpha=1$ $(\alpha\in W,\ x\in G)$ implies $\alpha=0$ or $x=1$.

Lemma 2. *Let* $f:\Re(G)\to\Re(G_1)$ *be an affine isomorphism between torsion-free w-power groups over the rings W and* W_1, *then*

(i) $Z(G_1)=Z(f(G))$;

(ii) *the nilpotency classes on the subgroups coincide;*

(iii) *there exists an isomorphism* $\sigma : W \to W_1$ *such that* $f\left(a^{\mu}\right)=\left[f\left(a\right)\right]^{\sigma(\mu)}$, $\mu \in W$, $a \in G$.

Theorem (Fundamental theorem of affine geometry for *w*-power groups). *Let* $f : \Re\left(G\right) \to \Re\left(G_1\right)$ *be an affine isomorphism between the w-power groups G and* G_1 *over the fields W and* W_1, *respectively, if G is nilpotent of class* 2 *then f is either a semilinear isomorphism or a semilinear antiisomorphism with respect to the isomorphism* $\sigma : W \to W_1$.

Remark 2. Thus we have concluded that using the coset lattices the fundamental theorem of affine geometry can be proved for *w*-power groups over fields, while the fundamental theorem of projective geometry is false [2, 6, 8, 9].

The example below shows that the theorem is false for the class of nilpotency ≥ 3.

Example. Let *G* be *w*-power group over the field *F* generated by the elements *a*, *b* and have the defining relations

$$1 \neq \left[a,b\right]=z_1, \quad \left[a,z\right]=\left[b,z\right]=1, \quad \left[a,z_1\right]=\left[b,z_1\right]=z.$$

It is clear that *G* is nilpotent of class 3 and $Z\left(G\right)=\left\langle\left[a,b\right]\right\rangle$. For an arbitrary element

$$l=a^{\alpha}b^{\beta}\left[a,b\right]^{\gamma}z^{\mu} \in G, \ \alpha,\beta,\gamma,\mu \in F$$

consider the map $f : G \to G$

$$f\left(l\right)=a^{\alpha}b^{\beta}\left(\left[a,b\right]z\right)^{\gamma}z^{\mu}.$$

We have

$$f\left(\left[a,b\right]\right)^{\alpha} f\left(\left[a,b\right]\right)^{\beta}=\left(\left[a,b\right]z\right)^{\alpha}\left(\left[a,b\right]z\right)^{\beta}=\left(\left[a,b\right]z\right)^{\alpha+\beta}=$$
$$=f\left(\left[a,b\right]\right)^{\alpha+\beta}=f\left(\left[a,b\right]^{\alpha+\beta}\right)=f\left(\left[a,b\right]^{\alpha}\left[a,b\right]^{\beta}\right).$$

So for any

$$l_1=a^{\alpha_1}b^{\beta_1}\left[a,b\right]^{\gamma_1}z^{\mu_1}, \ \alpha_1,\beta_1,\gamma_1,\mu_1 \in F,$$
$$l_2=a^{\alpha_2}b^{\beta_2}\left[a,b\right]^{\gamma_2}z^{\mu_2}, \ \alpha_2,\beta_2,\gamma_2,\mu_2 \in F.$$

Thus we have $f\left(l_1,l_2\right)=f\left(l_1\right)f\left(l_2\right)$, $f\left(l^{\alpha}\right)=\left[f\left(l\right)\right]^{\alpha}$. Therefore

$$f\left(l_1 \cup l_2\right)=f\left(l_1\left\langle l_1 l_2^{-1}\right\rangle\right)=f\left(l_1\right)\left\langle\left(l_1 l_2^{-1}\right)\right\rangle=f\left(l_1\right)\left\langle f\left(l_1\right)\left[f\left(l_2\right)\right]^{-1}\right\rangle=f\left(l_1\right)\cup f\left(l_2\right).$$

Consequently, *f* is an affine automorhism of the lattice $\Re\left(G\right)$, which is not a semilinear automorphism of *G*.

Remark 3. For a similar example for Lie algebra see [8].

Remark 4. From this example we can conclude that the theorems from [2, 9] need correction, i.e. they are valid for class of nilpotency ≤ 2.

Acknowledgements. This work was supported by Shota Rustaveli National Science Foundation (Projects # GNSF/ST08/3-383 and # GNSF/ST08/3-396).

მათემატიკა

ხარისხოვანი ჯგუფების აფინური მესერები და მათი იზომორფიზმები

თ. ბოკელავაძე*, ა. ლაშხი**

** აკაკი წერეთლის სახელმწიფო უნივერსიტეტი, ზუსტ და საბუნებისმეტყველო მეცნიერებათა ფაკულტეტი, ქუთაისი*

*** საქართველოს ტექნიკური უნივერსიტეტი, თეორიული ინფორმატიკისა და კომპიუტერული მოდელირების დეპარტამენტი, თბილისი*

ნაშრომში ჰოლის ხარისხოვანი G ჯგუფისათვის W რგოლზე იგება მოსაზღვრე კლასების $\Re(G)$ მესერი. ასეთ მესერებს ეწოდება G ჯგუფის აფინური ან მოსაზღვრე კლასების მესერი. რადგანაც $\Re(G)$ მესერში მხოლოდ G-ს ელემენტები ფარავენ ცარიელ სიმრავლეს, იზომორფიზმი $f:\Re(G)\to\Re(G_1)$ განსაზღვრავს ბიექციას $f:G\to G_1$. ყველა შესაძლო f იზომორფიზმებიდან ჩვენ ავარჩევთ ისეთებს, რომელთათვისაც $f(1)=1$. ასეთ იზომორფიზმებს ვუწოდებთ აფინურ იზომორფიზმებს. მტკიცდება შემდეგი თეორემა: ვთქვათ, $f:\Re(G)\to\Re(G_1)$ აფინური იზომორფიზმია ჰოლის w-ხარისხოვან G და G_1 ჯგუფებს შორის, რომლებიც განსაზღვრულნი არიან W და W_1 ველებზე, თუ G არის 2 კლასის ნილპოტენტური, მაშინ f ან ნახევრადწრფივი იზომორფიზმია ან ნახევრადწრფივი ანტიიზომორფიზმი, რომელიც შეესაბამება $\sigma:W\to W_1$ ველების იზომორფიზმს. იგება მაგალითი, რომელიც გვაჩვენებს, რომ თეორემა არ არის სამართლიანი, როდესაც ნილპოტენტურობის კლასი ≥ 3.

REFERENCES

1. *Ph. Hall* (1969), The Edmonton Notes on Nilpotent Groups. Queen Mary College Mathematics Notes. London.
2. *T. Bokelavadze, A. Lashkhi* (2006), Bull. Georg. Natl. Acad. Sci., **173**, 1: 17-18.
3. *A.A. Lashkhi* (1986), J. Algebra, **99**, 1: 80-88.
4. *R. Schmidt* (1994), Subgroup Lattices of Groups. Berlin, New York: de Gruyter.
5. *T. Gelashvili, A. Lashkhi* (1999), Proc. A. Razmadze Math. Inst. **119**: 43-58.
6. *A. Lashkhi* (2007), Doklady Akademii Nauk Rossii, **417**, 3: 313-315 (in Russian).
7. *T. Kvirikashvili, A. Lashkhi* (2004), Sovrem. Mat. Prilozh., No. 22, Algebra i Geom., 126-138 (in Russian); English transl.: (2006), J. Math. Sci., **137**, 5: 5161-5173 (N. Y.).
8. *A. Lashkhi* (2011), Bull. Georg. Natl. Acad. Sci. New Series, **5**, 1: 21-30.
9. *T. Bokelavadze, A. Lashkhi* (2009), Doklady Akademii Nauk Rossii, **429**, 6: 1-4 (in Russian).

Received May, 2011