

Physics

Quantum Properties of Scalar Fields with Broken $SU(2)$ –Symmetry

Vakhtang Gogokhia* and Avtandil Shurgai**

* *Research Institute for Particle and Nuclear Physics, Wigner Research Centre for Physics, Hungarian Academy of Sciences, Budapest, Hungary; Andria Razmadze Mathematical Institute of I. Javakhisvili Tbilisi State University, Tbilisi*

** *Andria Razmadze Mathematical Institute of I. Javakhisvili Tbilisi State University, Tbilisi*

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ABSTRACT. We consider the simplest theoretical model of scalar fields in one spatial dimension with an internal symmetry. We apply the Schrödinger picture to describe the quantum properties of localized solutions. The use of the method of collective coordinates allows to develop a perturbation theory, which exactly describes the symmetry properties of the theory. As an example, we consider the theory of three scalar fields with a broken $SU(2)$ symmetry, for which we derive the dependence of the energy spectrum on a quantum number corresponding to a charge symmetry. The energy of quantum bound states is the classical energy plus the energy of quantum excitations. The result does not depend on the specific form of the interaction potential. © 2012 Bull. Georg. Natl. Acad. Sci.

Key words: *collective coordinates, charge symmetry, localized field configurations.*

Quantization of localized in space solutions has a long history. There are several approaches to this problem, mainly within the framework of a functional integration [1-8]. The method of collective coordinates, which indeed uses the parameters of a symmetry group as dynamical variables, is one of them. We apply in the present investigation the operator approach of this method for solving the Schrödinger equation. The advantage of this approach is the absence of an ambiguity related with the operator ordering. Besides, a perturbation theory can be constructed which is manifestly invariant with respect to the group of the symmetry of a theory under consideration in every order of the perturbation series.

We consider below the simplest model with internal symmetry, which is of interest even today, although many articles were devoted to this issue [3, 10-12]. The point is that it is possible in theories with internal symmetry to obtain stable field configurations in more than one space dimension. In papers [3, 10-12] the charge symmetry has been studied and contribution of the charge depending terms to the energy of the system has been evaluated. We want to focus our attention on [10]. In this article the charge symmetry is investigated in detail. It has been shown that there exist charged classical field configurations that are stable owing to the existence of a charge. The centrifugal term and its contribution to the energy have been calculated in the framework of the functional integration. The charge is *assumed* to take on integer values.

The approach used in the present article starts from the quantum theory and suggests the construction of a perturbation theory around the classical neutral field configuration (around charged configurations is also possible). The charge is quantized such that the exact dependence of the energy on the charge is represented. Besides, the operator ordering problem, which requires special treatment in the frame of a functional integration, does not arise in this approach.

The Model with the $U(1)$ Symmetry

We consider the model of scalar triplet in one space dimension with a broken $SU(2)$ symmetry. The Hamiltonian of the model is

$$H = \int dx \{ \pi_\alpha(x) \pi_\alpha(x) + \frac{1}{2} \frac{\partial \varphi_a(x)}{\partial x} \frac{\partial \varphi_a(x)}{\partial x} + U(\varphi_i(x), \varphi_3(x), g) \} \quad (1)$$

with the following commutation relations between the field operators:

$$[\varphi_\alpha(x), \pi_\beta(y)] = i \delta_{\alpha\beta} \delta(x-y). \quad (2)$$

The Greek indices α, β, γ take on values 1, 2, 3 and Latin indices $-1, 2$. The fields $\varphi_\alpha(x)$ form the charged and neutral fields. The potential $U(\varphi_i(x), \varphi_i(x), \varphi_3(x), g)$ is suggested to be breaking the $SU(2)$ symmetry and obeys the condition:

$$U(\varphi_i(x), \varphi_i(x), \varphi_3(x), g) = g^2 U(g\varphi_i(x), g\varphi_3(x), 1). \quad (3)$$

Thus the model is $U(1)$ invariant and, as a result, the charge of the system

$$Q = \int dx \{ \varphi_1(x) \pi_2(x) - \varphi_2(x) \pi_1(x) \} \quad (4)$$

is conserved. We apply the method of collective coordinates to the Schrödinger equation

$$H\Psi(\varphi_i(x), \varphi_3(x)) = E\Psi(\varphi_i(x), \varphi_3(x)) \quad (5)$$

Mention that the operator $\pi_\alpha(x)$ is considered as $-\partial/i\partial\varphi_\alpha(x)$.

Let us introduce the transformations

$$\varphi_i(x) = D_{ij}(\mathcal{G}) [g\delta_{1j}u(x) + \Phi_j(x)], \quad \varphi_3(x) = g\sigma(x) + \Phi_3(x) \quad (6)$$

in which the function $u(x)$ is a c-number and the parameter \mathcal{G} together with the fields $\Phi_\alpha(x)$ compose a new set of operators. The matrix $D_{ij}(\theta)$ is a two-dimensional matrix of rotation:

$$D_{ij}(\mathcal{G}) = \begin{pmatrix} \cos \mathcal{G} & -\sin \mathcal{G} \\ \sin \mathcal{G} & \cos \mathcal{G} \end{pmatrix}. \quad (7)$$

In order to retain the total number of independent variables a subsidiary condition has to be imposed, namely

$$\int dx N(x) \Phi_2(x) = 0. \quad (8)$$

The c-number $N(x)$ is normalized as follows:

$$\int dx N(x)u(x) = 1. \tag{9}$$

We must now express the momenta $\pi_\alpha(x)$ in terms of the new variables $\theta, \Phi_\alpha(x)$.

This is done according to the ordinary rules of differentiation provided that the subsidiary condition (8) is taken into account. In addition to (8) we need to introduce the following projection operator $A_{ik}(x, y) = \delta_{ik}\delta(x-y) - \delta_{2i}\delta_{2j}N(x)u(y)$ with the properties:

$$\int dy A_{i2}(x, y)N(y) = \int dx u(x)A_{2j}(x, y) = 0, \quad \int dy A_{ij}(x, y)A_{jk}(y, z) = A_{ik}(x, z), \tag{10}$$

The fields $\Phi_\alpha(x)$ can be expressed as

$$\Phi_k(x) = \int dz A_{ij}(y, z)\bar{D}_{ji}(\mathcal{G})(\delta_{i1}u(z) - \Phi_i(z)) \tag{11}$$

in which $\bar{D}(\theta) = D^{-1}(\theta)$. Taking the functional derivative of (8) and (11) with respect to $\varphi_\alpha(x)$, we obtain:

$$\pi_k(x) = \bar{D}_{ji}(\mathcal{G})\{\Pi_j(x) + \frac{N(x)\delta_{2k}}{g-F}(p_g + \bar{Q})\}, \tag{12}$$

$$\pi_3(x) \equiv \Pi_3(x) = \frac{\delta}{i\delta\Phi_3(x)}. \tag{13}$$

In this expression the following notations are used: $\bar{Q} = \bar{J}_{ik} \int dx \Phi_k(x)\Pi_i(x)$ is the charge of mesons (described by the fields $\Phi_j(x)$), the antisymmetric matrix

$$\bar{J}_{ik} = \frac{\partial \bar{D}(\mathcal{G})}{\partial \mathcal{G}} \Big|_{\mathcal{G}=0}$$

is the element of the algebra of the group $U(1)$, $\Pi_i(x) = \int dy A_{ji}(y, x) \left(\frac{\delta}{i\delta\Phi_j(y)} \right)$, $F = \int dx N(x)\Phi_1(x)$. The

new set of the variables of the system obey the following nonzero commutation relations:

$$[\mathcal{G}, p_g] = i, \quad [\Phi_i(x), \Pi_k(y)] = iA_{ik}(x, y), \quad [\Phi_i(x), \Pi_k(y)] = i\delta(x-y). \tag{14}$$

We can now write down the kinetic energy in the new representation:

$$K = \int dx \frac{1}{2} \pi_a(x)\pi_a(x) = \int dx \left\{ \frac{1}{2} \Pi_3^2(x) + [\Pi_k(x) + \frac{N(x)\delta_{2k}}{g-F}(p_g + \bar{Q})][\Pi_k(x) + \frac{N(x)\delta_{2k}}{g-F}(p_g + \bar{Q})] - i \frac{N(x)}{g-F} \Pi_1(x) \right\}. \tag{15}$$

As we can see the kinetic energy does not contain the variable \mathcal{G} . It is easy to verify that in the new representation the charge operator Q is reduced to p_g , so $Q = p_g$. Now, the equality (3) and the transformations (6) make it possible to expand the potential U in a series by inverse powers of g , namely

$$\begin{aligned}
 U = & g^2 U(u(x), \sigma(x)) + g \frac{\partial U(u(x), \sigma(x))}{\partial u(x)} + g \frac{\partial U(u(x), \sigma(x))}{\partial \sigma(x)} + \\
 & \frac{1}{2} \frac{\partial^2 U(u(x), \sigma(x))}{\partial u(x)^2} + \frac{1}{2} \frac{\partial^2 U(u(x), \sigma(x))}{\partial \sigma(x)^2} + \frac{\partial^2 U(u(x), \sigma(x))}{\partial u(x) \partial \sigma(x)} + \dots
 \end{aligned} \quad (16)$$

Thus the variable \mathcal{G} is cyclic and the operator $p_{\mathcal{G}}$ can be replaced with the c-number. Before doing this, we introduce the transformations, which eliminate the linear and cross terms in the operators $\Phi_j(x)$ and $\Pi_j(x)$ from the kinetic energy. First of all we assume $N(x) = \lambda u(x)$ such that owing to (9) $\lambda \int dx u^2(x) = 1$. The quantity F can now be rewritten as λh with $h = \int dx u(x) \Phi_1(x)$. Secondly, we change the wave functional $\Psi(\theta, \Phi_{\alpha}(x))$ as follows:

$$\Psi(\mathcal{G}, \Phi_{\alpha}(x)) = \frac{1}{\sqrt{g - \lambda h}} \widetilde{\Psi}(\mathcal{G}, \Phi_{\alpha}(x)).$$

Making use of this substitution we finally obtain for the Hamiltonian

$$\begin{aligned}
 H = & \int dx \left\{ \frac{1}{2} \frac{\partial}{\partial x} (\delta_{i1} u(x) + \Phi_1(x)) \frac{\partial}{\partial x} (\delta_{i1} u(x) + \Phi_1(x)) + \right. \\
 & \frac{1}{2} \frac{\partial}{\partial x} (\sigma(x) + \Phi_3(x)) \frac{\partial}{\partial x} (\sigma(x) + \Phi_3(x)) + U + \frac{1}{2} \Pi_{\alpha}(x) \Pi_{\alpha}(x) \left. \right\} + \\
 & \frac{\lambda}{2} \left[\frac{1}{g - \lambda h} (p_{\mathcal{G}} + \bar{Q}) - \frac{1}{g - \lambda h} (p_{\mathcal{G}} + \bar{Q}) - \frac{1}{4} \right]
 \end{aligned} \quad (17)$$

which acts on $\widetilde{\Psi}(\mathcal{G}, \Phi_{\alpha}(x))$. We can now factor out the \mathcal{G} -dependence in the wave functional since \mathcal{G} is cyclic:

$$\widetilde{\Psi}(\mathcal{G}, \Phi_{\alpha}(x)) = \exp(im\mathcal{G}) \Psi'(\Phi_{\alpha}(x))$$

with $m = 0, \pm 1, \pm 2, \pm 3, \dots$. We now replace the operator $p_{\mathcal{G}}$ by m . All calculations are so far accurate and we did not make any approximation. We have eliminated $p_{\mathcal{G}}$ -dependence from the Hamiltonian and it is now the function of the quantum number m , thereby the charge has been quantized. One can now construct a perturbation theory (which will be manifestly $U(1)$ -invariant in every order) by means of expanding the Hamiltonian, the energy and the $\Psi'(\Phi_{\alpha}(x))$ in terms of inverse powers of g as follows:

$$\begin{aligned}
 H = & g^2 H_0 + g H_1 + H_2 + g^{-1} H_3 + g^{-2} H_4 + \dots \\
 E = & g^2 E_0 + g E_1 + E_0 + g^{-1} E_3 + g^{-2} E_4 + \dots \\
 \Psi'(\Phi_{\alpha}(x)) = & \Psi_1 + g \Psi_2 + g^2 \Psi_2 + \dots
 \end{aligned} \quad (18)$$

We next solve the system of equations:

$$\begin{aligned}
 (H_0 - E_0) \Psi_0 &= 0, \\
 (H_0 - E_0) \Psi_1 + (H_1 - E_1) \Psi_0 &= 0, \\
 (H_0 - E_0) \Psi_2 + (H_1 - E_1) \Psi_1 + (H_2 - E_2) \Psi_0 &= 0, \\
 \dots
 \end{aligned} \quad (19)$$

We do not specify the potential U since we aim to quantize the charge and to obtain the exact dependence of the Hamiltonian on the charge. We shall only make some remarks regarding the equations (19). It is evident, that the leading equations of order g^2 and g reproduce the classical equations of motion with the zero charge solutions $u(x)$ and σ and classical energy of neutral field configurations. The equation of order g^0 is bilinear in field operators $\Phi_\alpha(x)$ and $\Pi_\alpha(x)$, which formally can be diagonalized. The result is infinite sum of oscillators and must be regularized. The charge dependence of the energy arises in the approximation of the order g^{-2} and is of the form:

$$E_4 = \frac{\lambda}{2} (m^2 + 2m \langle \bar{Q} \rangle_0 + \langle \bar{Q}^2 \rangle_0 - \frac{1}{4}).$$

The symbol $\langle \rangle_0$ denotes the average over the ground states of oscillators. One can show that $\langle \bar{Q} \rangle_0 = 0$. We see that the charge is quantized in a natural way and takes on integer values as a consequence of the periodicity of the wave functional in \mathcal{G} (and this is an exact result, which does not depend on any further approximation) as opposed to the article [10], in which the charge is *assumed* to be an integer.

In this article we were interested in the simplest case of internal symmetry. The realistic theories with higher symmetry in more than 2-dimensions are of more interest which is the subject of further investigation. Besides, application of our method to QCD at finite temperature [13, 14] is of great interest. We suggest studying the topological properties of the QCD ground state and phasing transitions due to the objects like periodic instantons.

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ფიზიკა

სკალარული ველების კვანტური თვისებები დარღვეული $SU(2)$ სიმეტრიით

ვ. გოგოხია*, ა. შურგაია**

* უნგრეთის მეცნიერებათა აკადემია, ვიგნერის ფიზიკის კვლევითი ცენტრი, ნაწილაკთა და ბირთვული ფიზიკის კვლევითი ინსტიტუტი, ბუდაპეშტი, უნგრეთი; თბილისის ი. ჯავახიშვილის სახ. სახელმწიფო უნივერსიტეტი, ა. რაზმაძის სახ. მათემატიკის ინსტიტუტი

** თბილისის ი. ჯავახიშვილის სახ. სახელმწიფო უნივერსიტეტი, ა. რაზმაძის სახ. მათემატიკის ინსტიტუტი

(წარმოდგენილია აკადემიის წევრის ა. ხელაშვილის მიერ)

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