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*Physics*

# Quantized Chiral Soliton as an Alternative to the Skyrme Model and γNΔ Vertex

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**ABSTRACT. The problem of soliton stability in the chiral sigma model is investigated. After the brief introductory relations the relevant expression for the Hamiltonian appropriate to the quantization of rotational and vibrational modes as well as the pion mass term is presented. Corresponding equation of motion is exhibited, from which the effective potential is extracted. The existence of minimum of this potential is established and the energy of the ground state is found. It is shown that stability may be achieved only after the breaking of chiral symmetry by pion mass term. The profile function is constructed which guarantees the stability. Mass spectra of nucleon and its resonances are derived. This is analogous to WKB quantization formula.**

$$E_n = m_\pi \left(\frac{c}{4a}\right)^{1/2} \left[W_{eff}(Y_0) + \sqrt{2W''_{eff}(Y_0)}\,(n+1/2)\right]$$

**where expressions for entering here parameters are given in the main text, Eqs. (15).**

**Unfortunately the profile functions applied do not satisfy to the equation of motion, therefore their status is unclear. Nevertheless the existence of minimum is a remarkable property for this problem and it underlines the importance of quantum consideration. Therefore it is interesting to calculate appropriate profile function. In this purpose the calculation scheme is proposed for the profile function in the Skyrme anzatz, which then may be applied to delta – nucleon transition problem. The most profound task is a reconstruction of the profile function from the integral relations. For the present only phenomenological consideration shows that by choosing of appropriate test profile functions rather good description of nucleon resonance masses may be achieved.** *© 2012 Bull. Georg. Natl. Acad. Sci.*

***Key words***: *chiral soliton, Skyrme model, Skyrme anzatz, mass formula, nucleon-delta transition form factors.*

Skyrme model [1] is a possible effective field theory for quantum chromodynamics (QCD). It explains many static properties of Baryons – nucleons and its resonances [2].

From the view of global symmetries, the Skyrme model is based on $SU(2)\times SU(2)$ chiral sigma algebra and its effective Lagrangian

$$L_C = \frac{F_\pi^2}{2} Tr\left[\partial_\mu U^+ \partial^\mu U\right], \qquad U \subset SU(2)\times SU(2) \tag{1}$$

As is well known [2] this Lagrangian does not provide for the stability of static solitonic solutions. It was found that to ensure stability the addition of various terms to Lagrangian (1) is necessary. But all of this is valid in the framework of classical field theory.

One of the widespread methods consists in addition of a new, so-called Skyrme term [1 ], which looks like this

$$L_{Sk} = \frac{1}{32g^2} Tr\left[U^+\partial_\mu U, U^+\partial_\mu U\right]^2 = \frac{\varepsilon^2}{4} Tr\left[L_\mu, L_\nu\right]^2, \tag{2}$$

where

$$L_\mu \equiv U^+\partial_\mu U, \tag{3}$$

by means of which the topological current may be constructed as

$$J_\mu = \frac{1}{24\pi^2} \varepsilon^{\mu\nu\lambda\rho} Tr\left(L_\nu L_\lambda L_\rho\right). \tag{4}$$

The topological charge is

$$Q = \frac{1}{24\pi^2} \varepsilon^{0\nu\lambda\rho} \int d^3x Tr\left(L_\nu L_\lambda L_\rho\right) \tag{5}$$

and it equals the winding number.

Skyrme [1] introduced the so-called "hedgehog" anzatz

$$U_0 = \exp\left\{i\boldsymbol{\tau}\cdot\boldsymbol{x}F(r)\right\}. \tag{6}$$

Where the profile function $F(r)$ must be imposed by definite boundary conditions. Namely, at infinity $U$ has to turn into the unit matrix, or $F(\infty) = 2\pi n$. In particular, $F(\infty) = 0$ is also permissible.

On the other hand, at the origin $F(r)$ must be a well-defined function and must provide turning to unit matrix of $U$ independently of direction, in which $r = 0$ point is approached. It means that we must have

$$\sin F(0) = 0, \quad i.e. \quad F(0) = n\pi \tag{7}$$

Accordingly, $Q = n$. The Skyrme term together with chiral one gives the following scale property for the energy functional

$$E = \frac{1}{\lambda} E^{(2)} + \lambda E^{(4)}, \qquad (x \to \lambda x) \tag{8}$$

and provides a minimum when $E^{(2)} = E^{(4)}$. At the same time the Energy is bounded from below by topological charge

$$E > 12\sqrt{2}\pi^2 \varepsilon F_\pi |Q|. \tag{9}$$

Therefore corresponding configurations are stable.

With the aid of chiral angle $F(r)$ the energy (or mass) of static configuration is expressed as

$$M_{Ch} = 2\pi F_\pi^2 \int_0^\infty drr^2 \left[\left(\frac{\partial F}{\partial r}\right)^2 + \frac{2}{r}\sin^2 F\right], \tag{10}$$

from which in accordance with the variation principle the equation of motion follows

$$r^2 \frac{d^2 F}{dr^2} + 2r \frac{dF}{dr} = \sin 2F(r). \tag{11}$$

Explicit analytic solution of this equation is impossible; therefore one can investigate only the behavior at small and large distances accounting for corresponding boundary conditions and some symmetry properties, such as under the scale transformation

$$F(r) \to F\left(\frac{r}{\lambda}\right) \tag{12}$$

and under the translation $F(r \to 0) = \pi - \frac{r}{\lambda}$.

Numerical calculations, performed by us, show that all the solutions starting from the point $F(0) = \pi$ reach the same asymptotic value $\frac{\pi}{2}$ for different values of negative slope $dF/dr|_{r=0}$. However, it differs from the correct boundary behavior, mentioned above. Therefore we have proved by numerical analysis that there are no stable solitonic solutions.

To achieve stability of the chiral solution vector fields are often used as well [3] instead of Skyrme term. There are also many other generalizations of the Skyrme model [4].

One of attractive ideas was proposed in the 1990s [5]. This idea rests on the analogy with quantum mechanics, namely the hydrogen atom after quantization becomes stable, or the bottom of the classical well is a stable point, whereas in quantum mechanics the lowest level is non-trivial according to Heisenberg's uncertainty principle, and so on.

The natural question arises: if *something like this can occur in the chiral model?* It follows from our discussion below that in certain circumstances stability may be guaranteed. Not restricting generality, we consider chiral Lagrangian, taking into account quantization of rotational and vibrational modes and the mass term of the pion.

Quantization is performed by using the transformed chiral field $F\left(\frac{r}{\lambda}\right)$ and accounting for $\lambda$ as a dynamical variable, i.e. $\lambda = \lambda(t)$. After this the matrix $U$ is taken as follows

$$U = A(t) U_0 \left(\frac{r}{\lambda(t)}\right) A^{-1}(t), \tag{13}$$

where $U_0$ is a Skyrme matrix (6) and $A(t)$ accomplishes quantization of rotational modes. Together with the pion's mass term we derive the following Schrödinger equation [6]

$$\left[-\frac{d^2}{dZ^2} + \left(\frac{b}{4a}\right)^{1/3} Z^{2/3} + \left(\frac{3}{4} + \frac{2a}{I} T(T+1)\right)\frac{1}{Z^2} + \frac{m_\pi^2}{F_\pi^2}\frac{c}{4}\frac{{}^2Z}{a}\right]\Phi = \frac{E}{F_\pi}\Phi, \tag{14}$$

where $Z = \sqrt{4a}\lambda^{3/4}$ and $\Phi$ is related to the true Schrödinger wave function $\Psi$ according to the relation $\Phi = \Psi \lambda^{3/4}$. Other quantities are defined by

$$a=\frac{8}{9}\pi\int_0^\infty dyy^4\left(\frac{dF}{dy}\right)^2;\quad b=2\pi\int_0^\infty dyy^2\left[\left(\frac{dF}{dy}\right)^2+\frac{2}{y^2}\sin^2 F\right], \tag{15a}$$

$$c=8\pi\int_0^\infty dyy^2\sin^2\frac{F}{2};\quad I=\frac{8\pi}{3}\int_0^\infty dyy^2\sin^2 F\ . \tag{15b}$$

As the effective potential for this equation is

$$W_{eff}\left(Z\right)=\alpha Z^{2/3}+\frac{\beta}{Z^2}+\gamma Z^2, \tag{16}$$

where

$$\alpha=\left(\frac{b^3}{4a}\right)^{1/3};\quad \beta=\frac{3}{4}+\frac{2a}{I}T(T+1);\quad \gamma=\frac{m_\pi^2}{4F_\pi^2}\frac{c}{a} \tag{17}$$

This potential has a non-trivial minimum when one of the parameters $\alpha$ or $\gamma$ is non-vanishing. Scale transformations now are

$$F\left(\frac{r}{\lambda}\right)=\lambda^3 F\left(r\right) \tag{18}$$

and $\alpha$ and $\gamma$ are not changed in this case. Let us extract $\gamma$ parameter from the equation by transformation $Y=\gamma^{1/4}Z$ . Then the Schrödinger equation gets the form

$$\left[-\frac{d^2}{dY^2}+\alpha\gamma^{-2/3}Y^{2/3}+\frac{\beta}{Y^2}+Y^2\right]\Phi\left(Y\right)=\frac{E}{\gamma^{1/2}F_\pi}\Phi\left(Y\right) \tag{19}$$

and the ground state energy is written as

$$E=\gamma^{1/2}F_\pi\varsigma\left(\beta,\alpha\gamma^{-2/3}\right) \tag{20}$$

The effective potential has a minimum, which may be determined from the following equation

$$\frac{\beta}{Y_0^2}=\frac{\alpha\gamma^{-2/3}Y_0^{2/3}}{3}+Y_0^3\ . \tag{21}$$

The solution of this equation looks like this

$$Y_0^{4/3}=\left(-\frac{q}{2}+\sqrt{Q}\right)^{1/3}+\left(-\frac{q}{2}-\sqrt{Q}\right)^{1/3}, \tag{22}$$

where

$$-\frac{q}{2}=\frac{\beta}{2}-\frac{\left(\alpha\gamma^{-2/3}\right)^3}{9},\qquad Q=\left(\frac{\beta}{2}\right)^2-\frac{\beta\left(\alpha\gamma^{-2/3}\right)^3}{9}. \tag{23}$$

The energy spectrum in WKB approximation is

$$E_n=m_\pi\left(\frac{c}{4a}\right)^{1/2}\left[W_{eff}\left(Y_0\right)+\sqrt{2W''_{eff}\left(Y_0\right)}\left(n+1/2\right)\right]. \tag{24}$$

Here

$$W_{eff}\left(Y_0\right)=\frac{4}{3}\alpha\gamma^{-2/3}Y_0^{2/3}+2Y_0^2\,, \tag{25a}$$

$$W''_{eff}\left(Y_0\right)=\frac{16}{9}\alpha\gamma^{-2/3}Y_0^{-2/3}+8\,. \tag{25b}$$

Because $\beta$ is positive, it follows that $Y_0>0$ and $W_{eff}\left(Y_0\right)>0,\ W''_{eff}\left(Y_0\right)>0$. Therefore limitation from below is guaranteed! Minimum of energy is achieved at $\alpha=0$. In this case

$$E_n=\frac{m_\pi}{2}\left(\frac{c}{a}\right)^{1/2}\left(4n+2+2\sqrt{\beta}\right)=\frac{m_\pi}{2}\left(\frac{c}{a}\right)^{1/2}\left(4n+\sqrt{3+8\frac{a}{I}T\left(T+1\right)}\right). \tag{26}$$

The analytic solution of the Schrödinger equation gives practically the WKB form

$$E_n=\frac{m_\pi}{2}\left(\frac{c}{a}\right)^{1/2}\left(4n+2+\sqrt{4+8T(T+1)\frac{a}{I}}\right)=\frac{m_\pi}{2}\left[\left(4n+2\sqrt{\frac{c}{a}}+\sqrt{\frac{4c}{a}+8T\left(T+1\right)\frac{c}{I}}\right)\right]. \tag{27}$$

Estimation gives

$$\frac{c}{I}=3\frac{\int dyy^2\sin^2\frac{F}{2}}{\int dyy^2\sin^2 F}=\frac{3}{4}\frac{\int dyy^2\sin^2\frac{F}{2}}{\int dyy^2\sin^2\frac{F}{2}\left(1-\sin^2\frac{F}{2}\right)}\geq\frac{3}{4} \tag{28}$$

This leads to the following inequality

$$E_0\geq m_\pi\sqrt{\frac{3T(T+1)}{2}}\,. \tag{29}$$

It follows that in the chiral soliton stabilization a key role belongs to the chiral symmetry breaking by pion mass term and to the quantized rotational modes.

The above-obtained results can be summarized as follows:
The soliton mass is given by

$$M=M_{Ch}+E_{n,T}\,, \tag{30}$$

where

$$E_{n,T}=\frac{m_\pi}{2}\sqrt{\frac{c}{a}}\left(4n+2+\sqrt{3+\frac{8a}{I}T(T+1)}\right);\qquad (n=0,1,...) \tag{31}$$

This is a mass formula for the nucleon $(T=1/2)$ and the delta resonances $\left(T=3/2\right)$.

We have elaborated these relations by numerical methods, using the following phenomenological profile functions

$$F_1(r)=\frac{2\pi}{1+\exp\left(0{,}15r\right)};\qquad F_2(r)=\frac{\pi}{1+r+0{,}25r^2} \tag{32}$$

The obtained results are given in the Table

| mass(GeV) | N | $N'$ | $N''$ | $\Delta$ | $\Delta'$ | $\sqrt{\left(r^2\right)}$ fm |
|---|---|---|---|---|---|---|
| exp | 0.94 | 1.440 | 1.710 | 1.232 | 1.600 | 0.72 |
| $F_1$ | input | 1.340 | 1.700 | 1.220 | 1.580 | 0.39 |
| $F_2$ | input | 1.320 | 1.660 | 1.240 | 1.580 | 0.40 |

We see that these test functions describe experimental mass spectra sufficiently well with only one input parameter used – the nucleon mass, while they do not satisfy the equation of motion. Therefore their status is unclear.

It is interesting to clarify how these model-functions can be applied to other problems. Especially, in calculations of electromagnetic form-factors of nucleon resonances. We consider below delta to nucleon transition form-factors. A relevant electric form-factor of this problem is defined for spacelike momentum transfer $q^2 > 0$ as a Fourier transform of its electric charge

$$G_E^p\left(-q^2\right) = \frac{1}{2}\int_0^\infty dr B_0\left(r\right) j_0\left(qr\right) \tag{33}$$

with
$$B_0\left(r\right) = -\frac{2}{\pi}\sin^2 F\left(\frac{dF}{dr}\right). \tag{34}$$

It is evident from this relation that one needs profile functions of our model. Therefore we give a scheme for elaboration profile function founded on model relations derived above. This scheme is as follows: Our main relations are (30) and (31) for the masses of an isodublet $\left(N, \Delta\right)$ and their radial excitations. We have for the mass difference

$$N' - N = 2m_\pi\sqrt{\frac{c}{a}} \tag{35}$$

From experimental data one can determine the value of $\sqrt{c/a}$ . Then construct the expression

$$N + N' = 2M_{Ch} + 4m_\pi\sqrt{\frac{c}{a}} + m_\pi\sqrt{\frac{c}{a}}f_T\left(\frac{a}{I}\right), \tag{36}$$

where
$$f_T\left(\frac{a}{T}\right) \equiv \sqrt{3 + 8\frac{a}{I}T(T+1)} . \tag{37}$$

This function may be evaluated from the difference

$$\Delta - N = E_{0,3/2} - E_{0,1/2} = \left[f_{3/2}\left(\frac{a}{I}\right) - f_{1/2}\left(\frac{a}{I}\right)\right]\frac{m_\pi}{2}\sqrt{\frac{c}{a}} \tag{38}$$

Therefore from masses of $N, N'$ and $\Delta$ one can calculate unknown quantities

$$\sqrt{\frac{c}{a}}, \qquad \frac{a}{I} \qquad and \qquad f_{n,T}\left(\frac{a}{I}\right)$$

After that we determine the chiral mass, $M_{Ch}$ . According to equation (10) we then restore the profile function $F\left(r\right)$ , and then we'll have all ingredients for calculating other physical quantities, among them $N\Delta\gamma$ transition form-factors.

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*ფიზიკა*

# დაკვანტული კირალური სოლიტონი, როგორც სკირმის მოდელის ალტერნატივა და $\gamma N\Delta$ წვერო

**ა. ხელაშვილი**

*აკადემიის წევრი, ი.ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის მაღალი ენერგიების ინსტიტუტი; საქართველოს საპატრიარქოს წმინდა ანდრია პირველწოდებულის სახ. ქართული უნივერსიტეტი, თბილისი*

**ნაშრომში შესწავლილია კირალური სოლიტონის სტაბილურობის საკითხები მისი ბრუნვითი და ვიბრაციული მოდების დაკვანტვის შედეგად. მცირეოდენი შესავალი თანაფარდობების შემდეგ მოცემულია ჰამილტონიანის გამოსახულება, რომელიც შეესაბამება ბრუნვითი და ვიბრაციული მოდების დაკვანტვას პიონის მასურ წევრთან ერთად. მოყვანილია სათანადო განტოლებები, საიდანაც აღდგენილია ეფექტური პოტენციალი. დადგენილია ამ პოტენციალისთვის მინიმუმის არსებობა და ნაპოვნია ძირითადი მდგომარეობის ენერგია. ნაჩვენებია, რომ სტაბილურობა მიიღწევა მხოლოდ კირალური სიმეტრიის დარღვევის შედეგად პიონის მასური წევრით. აგებულია პროფილური ფუნქცია, რომელიც უზრუნველჰყოფს სტაბილურობას. მიღებულია ნუკლონისა და მისი რეზონანსების მასური სპექტრი, რომელიც ანალოგიურია კვაზიკლასიკური დაკვანტვის ფორმულისა:**

$$E_n = m_\pi \left( \frac{c}{4a} \right)^{1/2} \left[ W_{eff}\left(Y_0\right) + \sqrt{2W''_{eff}\left(Y_0\right)}\left(n + 1/2\right) \right]$$

**აქ შემავალი სიდიდეების ცხადი გამოსახულებანი მოცემულია მთავარ ტექსტში (15). სამწუხაროდ, გამოყენებული სპექტრალური ფუნქციები არ აკმაყოფილებენ მოძრაობის განტოლებას, რის გამოც მათი სტატუსი ჯერჯერობით ნათელი არ არის. მიუხედავად ამისა, მინიმუმის არსებობა წარმოადგენს მნიშვნელოვან თვისებას ამ პრობლემაში. ამიტომაც სათანადო პროფილური ფუნქციის გამოთვლა საინტერესო ამოცანას წარმოადგენს. ამ მიზნით გადმოცემულია პროფილური ფუნქციის გამოთვლის შესაძლო სქემა სკირმის ჩასმისათვის, რომელიც შემდგომ შეიძლება გამოვიყენოთ დელტა-ნუკლონში გადასვლის ამოცანაში. პროფილური ფუნქციის აგება ინტეგრალური თანაფარდობებიდან წარმოადგენს ყველაზე რთულ პრობლემას. ამ მიზნით ჩვენ ჩავატარეთ მხოლოდ ფენომენოლოგიური განხილვა, რომელიც გვიჩვენებს, რომ გარკვეული პროფილური ფუნქციის მისადაგებით ვღებულობთ საკმარისად კარგ აღწერას ნუკლონური რეზონანსების მასებისათვის.**

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