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Mathematics

On the Pseudo-Maximal Likelihood Estimation of the Parameters of an Exponential Distribution by Grouped Observations with Censoring

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ABSTRACT. In the paper is studied the problem of estimating λ parameter of exponential distribution by the pseudo-maximum method for censoring observations. Existence and uniqueness of solution of the maximal likehood equation is shown. The solution is asymptotic consistence and effective estimator of λ parameter of exponential equation. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *pseudo-maximal likelihood, grouped observations, censoring incomplete observation.*

Let X be a random variable with the distribution function $F(x) = F(x,\theta)$ where $\theta \in \Theta$ is an unknown vector parameter in a finite-dimensional Euclidean space $\Theta \in R^q$. Assume that Θ is a compactum. We are to construct a consistent estimate θ using observations of the random variable X. The experiment is run so that we do not know the actual number of realizations and know only a part of these realizations.

Let the fixed points $-\infty \le t_1 \le t_2 \le \cdots \le t_k \le \infty$ be given on the straight line R (we do not exclude the case in which the first or the last point takes an infinite value). These points form intervals which can be of three categories:

(0) An interval (t_i, t_{i+1}) belongs to the zero category if in this interval we know neither individual values of the sampling nor the number of sample values of the random value X which occur in this interval.

(1) An interval (t_i, t_{i+1}) belongs to the first category if in this interval we do not know individual values of the sampling but know the number of sample values of the random variable X which occur in this interval. As usual this number will be denoted by n_i.

(2) An interval (t_i, t_{i+1}) belongs to the second category if in this interval we know individual values of the sampling.

We call a sampling of this type a partly grouped sampling with censoring.

The absence of information in the intervals of the zero category creates difficulties which we will try to overcome by assuming that we know the type of the distribution $F(x,\theta)$ and the number of sample values from the sampling which do not occur in the intervals of the zero category: $n = \sum_{(1),(2)} n_i$.

Let $A_i = (t_i, t_{i+1})$ be an interval of the zero category. Denote by m_i the number of sampling terms occurring in A_i. Then $r = n + \sum_{(0)} m_i$ is the total number of observations. Note that $\frac{m_i}{n + \sum_{(0)} m_i}$ is a relative frequency of occurrence of X in A_i.

If $\hat{F}_r(x)$ denotes an empirical distribution function, then

$$\frac{m_i}{n + \sum_{(0)} m_i} = \hat{F}_r(t_{i+1}) - \hat{F}_r(t_i) \tag{1}$$

and, according to the strengthened Bernoulli law of large numbers, it converges to $p_i(\theta) = F(t_{i+1},\theta) - F(t_i,\theta)$ with probability 1.

By summation of equalities (1) over all intervals of the zero category we find

$$\sum_{(0)} m_i = n \frac{\sum_{i\in(0)} \left[\hat{F}_r(t_{i+1}) - \hat{F}_r(t_i)\right]}{1 - \left[\sum_{i\in(0)} \left[\hat{F}_r(t_{i+1}) - \hat{F}_r(t_i)\right]\right]}.$$

Hence we obtain

$$m_i = n \frac{\hat{F}_r(t_{i+1}) - \hat{F}_r(t_i)}{1 - \left[\sum_{i\in(0)} \left[\hat{F}_r(t_{i+1}) - \hat{F}_r(t_i)\right]\right]}. \tag{2}$$

Let us apply the method of pseudo-maximal likelihood. Assume that the distribution density of the random variable X with respect to the Lebesgue measure is $f(x) = f(x,\theta)$. Then a likelihood function has the form

$$L_n(x;\theta) = \prod_{i\in(0)} \left[F(t_{i+1}) - F(t_i)\right]^{m_i} \prod_{i\in(1)} \left[F(t_{i+1}) - F(t_i)\right]^{n_i} \cdot \prod_{j=1}^{n_j} f(x_{je}), \tag{3}$$

where m_i, $i \in (0)$, are defined by formulas (2).

The finding of points of a maximum of the function $L(x;\theta)$ is complicated because the study of the smoothness properties of empirical functions is difficult. Therefore we consider the slightly corrected likelihood function

$$\overline{L}_n(x;\theta)=\prod_{i\in(0)}\left[F(t_{i+1})-F(t_i)\right]^{n\frac{F(t_{i+1})-F(t_i)}{1-\left[F(t_{i+1})-F(t_i)\right]}}\prod_{i\in(1)}\left[F(t_{i+1})-F(t_i)\right]^{n_i}\cdot\prod_{j=1}^{n_j}f(x_{je}). \quad (4)$$

Lemma [1]. *Let the following conditions be fulfilled:*

a) the distribution function $F(x,\theta)$ is continuous with respect to both variables and has the continuous derivative $f(x,\theta)=\frac{\partial F(x,\theta)}{\partial x}$;

b) the function $\overline{L}_n(x;\theta)$ has the absolute maximum $\theta=\overline{\theta}_n$.

Then $\overline{\theta}_n$ is an asymptotically consistent and asymptotically effective estimate of the true value of the parameter $\theta=\theta_0$.

Let X be an exponentially distributed random value with the density

$$f(x)=\begin{cases}\lambda e^{-\lambda x} & \text{if } x\ge 0\\ 0 & \text{if } x<0\end{cases}$$

where $\lambda>0$.

Assume that $[a,b]$ is an inaccessible interval for the observer and we do not know the number and individual observations in this zone. We, however, know the observations outside this interval: $X_1,X_2,\ldots,X_n$. It is required to estimate λ by means of these observations. For this, we use pseudo-maximal likelihood estimates.

To construct likelihood functions note that if we denote by k the number of terms from the general sampling which occur in $[a,b]$, then $\frac{k}{k+n}$ will be the frequency of occurrences in the interval $[a,b]$. Hence, by the Bernoulli-Kolmogorov theorem $\frac{k}{k+n}\to p$ a.s., where $p=F(b)-F(a)$

$$F(x)=\begin{cases}1-e^{-\lambda x} & \text{if } x\ge 0\\ 0 & \text{if } x<0\end{cases}.$$

Therefore, we define k as

$$\hat{k}=\frac{n\left[F(b)-F(a)\right]}{1-\left[F(b)-F(a)\right]}.$$

Since the probability that exactly k elements from the sampling will occur in the "black hole" is $\left[F(b)-F(a)\right]^k$, we can write the pseudo-likelihood function as follows:

$$L=\prod_{i=1}^{n}f(x_i;\lambda)\cdot\left[F(b)-F(a)\right]^{n\frac{F(b)-F(a)}{1-\left[F(b)-F(a)\right]}}=\lambda^n\cdot e^{-\lambda\sum_{i=1}^{n}X_i}\left(F(b)-F(a)\right)^{n\frac{F(b)-F(a)}{1-(F(b)-F(a))}}. \quad (5)$$

Note that in (5) the power may turn out to be a non-integer number, but it is always positive. Also note that in this formula the multiplication sign is used for the expression in front of the large bracket.

From (5) we write

$$\ln L = n\cdot\ln\lambda-\lambda\sum_{i=1}^{n}X_i+\frac{n\cdot\left(F(b)-F(a)\right)}{1-\left(F(b)-F(a)\right)}\cdot\ln\left(F(b)-F(a)\right),$$

$$\frac{d\ln L}{d\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}X_i+n\cdot\left(\frac{\left(b\cdot e^{-\lambda b}-a\cdot e^{-\lambda a}\right)\cdot\ln\left(F(b)-F(a)\right)\left(1-\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^2}+\right.$$

$$\left.+\frac{\left(b\cdot e^{-\lambda b}-a\cdot e^{-\lambda a}\right)\cdot\left(1-\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^2}+\frac{\left(F(b)-F(a)\right)\cdot\ln\left(F(b)-F(a)\right)\cdot\left(b\cdot e^{-\lambda a}-a\cdot e^{-\lambda b}\right)}{\left(1-\left(F(b)-F(a)\right)\right)^2}\right)=$$

$$=\frac{n}{\lambda}-\sum_{i=1}^{n}X_i+n\cdot\frac{\left(b\cdot e^{-\lambda b}-a\cdot e^{-\lambda a}\right)\cdot\left(1-\left(F(b)-F(a)\right)+\ln\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^2}\ . \tag{6}$$

Let us investigate expression (6) when $\lambda\to 0$. Note that when $\lambda\to 0$, we have $F(b)-F(a)=e^{-\lambda a}-e^{-\lambda b}\to 0$ and $\ln\left(F(b)-F(a)\right)\to-\infty$.

We obtain

$$\lim_{\lambda\to 0}\frac{d\ln L}{d\lambda}=\infty\ .$$

Indeed,

$$\lim_{\lambda\to 0}\frac{d\ln L}{d\lambda}=\lim_{\lambda\to 0}\frac{\dfrac{\left(1-\left(F(b)-F(a)\right)\right)^2}{n\cdot\left(b\cdot e^{-\lambda b}-a\cdot e^{-\lambda a}\right)\left(1-\left(F(b)-F(a)\right)+\ln\left(F(b)-F(a)\right)\right)}-\dfrac{\lambda}{n-\lambda\sum_{i=1}^{n}X_i}}{\dfrac{\left(1-\left(F(b)-F(a)\right)\right)^2\lambda}{n\cdot\left(b\cdot e^{-\lambda b}-a\cdot e^{-\lambda a}\right)\left(1-\left(F(b)-F(a)\right)+\ln\left(F(b)-F(a)\right)\right)\cdot\left(n-\lambda\sum_{i=1}^{n}X_i\right)}}=$$

$$=\lim_{\lambda\to 0}\frac{\left(1-\left(F(b)-F(a)\right)\right)^2\left(n-\lambda\sum_{i=1}^{n}X_i\right)-\lambda\cdot n\left(b\cdot e^{-\lambda b}-a\cdot e^{-\lambda a}\right)\left(1-\left(F(b)-F(a)\right)+\ln\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^2\cdot\lambda}=$$

$$=\lim_{\lambda\to 0}\frac{\left(1-\left(F(b)-F(a)\right)\right)^2\left(n-\lambda\sum_{i=1}^{n}X_i\right)}{\left(1-\left(F(b)-F(a)\right)\right)^2\cdot\lambda}-$$

$$-\lim_{\lambda\to 0}\frac{n\left(b\cdot e^{-\lambda b}-a\cdot e^{-\lambda a}\right)\left(1-\left(F(b)-F(a)\right)+\ln\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^2}=\infty.$$

Analogously, when $\lambda\to\infty$, we have

$$\lim_{\lambda\to\infty}\frac{d\ln L}{d\lambda}=-\infty .$$

The continuous function $\frac{d\ln L}{d\lambda}$ changes its sign and therefore there exists a point $\hat{\lambda}$ such that $\left.\frac{d\ln L}{d\lambda}\right|_{\lambda=\hat{\lambda}}=0$. Let us verify that at this point the second derivative is negative.

We write the second derivative in the form

$$\frac{d^2\ln L}{d\lambda^2}=-\frac{n}{\lambda^2}+\frac{n\cdot\left(a^2\cdot e^{-\lambda a}-b^2\cdot e^{-\lambda b}\right)\left(1-\left(F(b)-F(a)\right)+\ln\left(F(b)-F(a)\right)\right)\left(1-\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^3}+$$

$$+\frac{n\cdot\left(b\cdot e^{-\lambda b}-ae^{-\lambda a}\right)\left(-\left(b\cdot e^{-\lambda b}-ae^{-\lambda a}\right)\right)+\frac{b\cdot e^{-\lambda b}-ae^{-\lambda a}}{F(b)-F(a)}}{\left(1-\left(F(b)-F(a)\right)\right)^2}+$$

$$+\frac{2n\cdot\left(b\cdot e^{-\lambda b}-ae^{-\lambda a}\right)^2\left(1-\left(F(b)-F(a)\right)+\ln\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^3}. \tag{7}$$

Note that $x^2\cdot e^{-\lambda x}$, where $x>0$, is a decreasing function and therefore the first term and the second term in (7) are negative. Let us now show that the sum of the third term and the fourth term is negative. We have

$$\frac{2n\cdot\left(b\cdot e^{-\lambda b}-ae^{-\lambda a}\right)^2\left(1-\left(F(b)-F(a)\right)+\ln\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^3}+$$

$$+\frac{n\cdot\left(b\cdot e^{-\lambda b}-ae^{-\lambda a}\right)^2\left(\frac{1}{F(b)-F(a)}-1\right)\cdot\left(1-\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^3}=$$

$$=\frac{n\cdot\left(b\cdot e^{-\lambda b}-ae^{-\lambda a}\right)^2\left(2\left(1-\left(F(b)-F(a)\right)+\ln\left(F(b)-F(a)\right)\right)+\left(\frac{1}{F(b)-F(a)}-1\right)\cdot\left(1-\left(F(b)-F(a)\right)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^3}=$$

$$=\frac{n\cdot\left(b\cdot e^{-\lambda b}-ae^{-\lambda a}\right)^2\left(\frac{1}{F(b)-F(a)}-\left(F(b)-F(a)\right)+2\ln\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^3}. \tag{8}$$

If we take the equalities $\frac{1}{x}-x+2\ln x\le 0$, where $0<x\le 1$, into consideration, then we can show the negativity of relation (8), which means that $\hat{\lambda}$ is the unique point of a maximum of the likelihood equality.

The following theorem is valid by virtue of Lemma.

Theorem. *Suppose that we have a sampling of exponential random values* $X_1, X_2, \ldots, X_n$ *with an unknown parameter* λ. *The observation is carried out beyond the limits of the interval* $[a,b]$ *in which* *neither sampling terms nor their quantity are recorded. Then the estimate of a maximal pseudo-likelihood exists for* λ *and is the unique root of the equation*

$$\frac{n}{\lambda}-\sum_{i=1}^{n}X_i+\frac{n\cdot\left(b\cdot e^{-\lambda b}-ae^{-\lambda a}\right)\left(1-\left(F(b)-F(a)\right)+\ln\left(F(b)-F(a)\right)\right)}{\left(1-\left(F(b)-F(a)\right)\right)^2}=0\,.$$

Moreover, this estimate is asymptotically consistent and effective.

მათემატიკა

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(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

ნაშრომში შეფასების ფსევდომაქსიმალური მეთოდის გამოყენებით შესწავლილია ექსპონენციალური განაწილების λ პარამეტრის შეფასების ამოცანა ცენზურირებული დაკვირვებების საფუძველზე. ექსპონენციალური განაწილებისათვის ნაჩვენებია შესაბამისი ფსევდომაქსიმალური დასაჯერობის განტოლების ამონახსნის არსებობა და ერთადერთობა. ეს ამონახსნი წარმოადგენს ესპონენციალური განაწილების λ-პარამეტრის ასიმტოტურად ძალდებულ და ეფექტურ შეფასებას.

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