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Mathematics

Divergence of Fourier Series with Respect to Systems of Products of Bases

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ABSTRACT. It is proved that the feature of function systems being products of quite general type bases is almost everywhere divergence of certain blocks (and consequently, almost everywhere divergence in Pringsheim sense) of Fourier series in integral classes wider than $L(\ln^+ L)^{d-1}$. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *multiple Fourier series, basis, complete orthonormal system, Haar system, divergence.*

Definitions and notation. Let a sequence $\varepsilon = (E_k^{(i)})$ $(k \in \mathbb{N},\ 1 \le i \le 2^k)$ of measurable subsets of $\mathbb{I} \equiv [0,1]$ be such that $|E_k^{(i)}| = 1/2^k$, $E_k^{(i)} = E_{k+1}^{(2i-1)} \cup E_{k+1}^{(2i)}$ and $E_k^{(i)} \cap E_k^{(j)} = \varnothing$ $(i \ne j)$. The Haar type system $H^{(\varepsilon)} = (h_n^{(\varepsilon)})_{n\in\mathbb{N}}$ corresponding to the sequence ε is defined as follows (see e.g. [1, Ch. 3, §4]):

$$h_n^{(\varepsilon)} = 2^{k/2}\chi_{E_{k+1}^{(2i-1)}} - 2^{k/2}\chi_{E_{k+1}^{(2i)}} \qquad (n = 2^k + i,\quad k \in \mathbb{N},\quad 1 \le i < 2^k),$$

where χ_A denotes the characteristic function of a set A. For the case $E_k^{(i)} = \left(\frac{i-1}{2^k}, \frac{i}{2^k}\right)$ the construction gives the ordinary Haar system $H = (h_n)$.

Denote by $L^0(\mathbb{I})$ the space of all measurable functions on $\mathbb{I}$ and by P_Φ denote the class of all polynomials with respect to a system of functions $\Phi = (\varphi_n)$.

Following [2, Ch. III, §1] let us say that a Banach space $B \subset L^0(\mathbb{I})$ satisfies the (*)-condition if $B \supset P_H$ and for every $F \in B^*$ there is a function $\psi_F \in L(\mathbb{I})$ such that

$$F(f) = \int_{\mathbb{I}} f\psi_F \qquad \text{for every} \qquad f \in P_H .$$

Let (φ_n) be a basis in a Banach space $B \subset L^0(\mathbb{I})$ satisfying the (*)-condition and (F_n) be the system of functionals dual to (φ_n). (ψ_{F_n}) will be called the system of functions dual to (φ_n).

Let $\Phi^{(1)} = (\varphi_n^{(1)}), \dots, \Phi^{(d)} = (\varphi_n^{(d)})$ be systems of functions on $\mathbb{I}$. Their product $\Phi^{(1)} \times \cdots \times \Phi^{(d)}$ is defined as the system of functions

$$\varphi_n(x) = \varphi_{n_1}^{(1)}(x_1) \cdots \varphi_{n_d}^{(d)}(x_d),$$

where $n = (n_1, \dots, n_d) \in \mathbb{N}^d$, $x = (x_1, \dots, x_d) \in \mathbb{I}^d$. Denote by $\Phi^{(d)}$ the d-th power of a system Φ.

Let for each $1 \le k \le d$: B_k be a Banach space satisfying the (*)-condition, $\Phi^{(k)}$ be a basis in B_k, and let $\Psi^{(k)}$ be the system of functions dual to $\Phi^{(k)}$. Suppose $\Phi = (\varphi_n)_{n \in \mathbb{N}^d}$ and $\Psi = (\psi_n)_{n \in \mathbb{N}^d}$ be the products of the systems $\Phi^{(k)}$ and $\Phi^{(k)}$, respectively. For a function $f \in L(\mathbb{I}^d)$ with $f\psi_n \in L(\mathbb{I}^d)$ $(n \in \mathbb{N}^d)$ the Fourier coefficients and series of f with respect to the system Φ are defined as follows:

$$c_n(f, \Phi) = \int_{\mathbb{I}^d} f\psi_n, \qquad \sigma(f, \Phi) \sim \sum_n c_n(f, \Phi)\varphi_n,$$

respectively.

For a system of functions $\Phi = (\varphi_n)$ and a class $A \subset L^0(\mathbb{I})$ we will write $\Phi \subset A$ if $\varphi_n \in A$ for each n.

Below the convergence and divergence of multiple series will be understood in the Pringsheim sense.

Following [3, Ch. 1, §4], an Orlicz space $M(L)$ is said to satisfy the Δ'-condition if there exist constants $t_0 > 0$ and $c > 0$ such that $M(t_1 t_2) \le cM(t_1)M(t_2)$ if $t_1, t_2 > t_0$.

Result. Olevskii developed the method of expanding the singularities of the Haar system to arbitrary complete orthonormal systems and bases in function spaces. The method is based on the following lemma (see e.g. [2, Ch. III, §1] or [1, Ch. 9, §1]).

Lemma A. *Let $B \subset L^0(\mathbb{I})$ be a Banach space satisfying the (*)-condition and let $\Phi = (\varphi_n)$ be a basis in B. Then there exist a Haar type system $(h_k^{(\varepsilon)})$ and a sequence of indices $n_k \uparrow \infty$ such that each $h_k^{(\varepsilon)}$ is representable in the form*

$$h_n^{(\varepsilon)}(x) = \sum_{n=n_k+1}^{n_{k+1}} a_n \varphi_n(x) + \alpha_k(x) \qquad (x \in \mathbb{I}),$$

where

$$\|\alpha_k\|_B < \frac{1}{2^k},$$

$$\sum_{i=1}^{k-1} \sum_{j=k+1}^{\infty} \left\| \sum_{m=n_j+1}^{n_{j+1}} c_m(\alpha_i, \Phi)\varphi_m \right\|_B < \frac{1}{2^k}.$$

Various applications of Lemma A to the problems of convergence of one-dimensional Fourier series can be found in [2, Ch. III, §1] and [1, Ch. 9, §1].

It is known that (see [4, 5]) $L(\ln^+ L)^{d-1}(\mathbb{I}^d)$ is the widest integral class in which the almost everywhere convergence of Fourier-Haar series is provided. In particular, T. Zerekidze [5] proved

Theorem A. *Let* $M:[0,\infty)\to[0,\infty)$ *be an increasing function such that* $\lim_{t\to\infty}\frac{M(t)}{t(\ln t)^{d-1}}=0$. *Then there exists a function* $f\in M(L)(\mathbb{I}^d)$ *whose Fourier-Haar series diverges almost everywhere.*

G. Karagulyan [6] extended Theorem A to complete orthonormal double systems of the type $\Phi\times\Phi$ with $\Phi\subset L^\infty(\mathbb{I})$. Namely, in [6] the following statement is proved.

Theorem B. *Let* $\Phi\subset L^\infty(\mathbb{I})$ *be a complete orthonormal system and let* $f\in L(\mathbb{I}^2)$ *be a function whose Fourier-Haar series diverges almost everywhere. Then there exists an equimeasurable with f function g on* $\mathbb{I}^2$ *whose Fourier series with respect to the system* $\Phi\times\Phi$ *diverges almost everywhere.*

In [6], by Theorem A, from Theorem B was derived

Theorem C. *Let* $\Phi\subset L^\infty(\mathbb{I})$ *be a complete orthonormal system. Then for every increasing function* $M:[0,\infty)\to[0,\infty)$ *with* $\lim_{t\to\infty}\frac{M(t)}{t\ln t}=0$ *there exists a function* $f\in M(L)(\mathbb{I}^2)$ *whose Fourier series with respect to the system* $\Phi\times\Phi$ *diverges almost everywhere.*

The result of [7, 8] implies

Theorem D. *For every* $f\in L\setminus L(\ln^+ L)^{d-1}(\mathbb{I}^d)$ *there exists an equimeasurable with f function g on* $\mathbb{I}^d$ *such that a general term of the Fourier-Haar series of g diverges unboundedly almost everywhere, i.e.*

$$\overline{\lim_{n\to\infty}}\left|c_n(g,H^d)h_n(x)\right|=\infty$$

for almost every $x\in\mathbb{I}^d$, *where* $H^{(d)}=\left(h_n\right)_{n\in\mathbb{N}^d}$ *is a d-tuple Haar system. Consequently, the Fourier-Haar series of g diverges almost everywhere.*

Note that before [7] R. Getsadze [9] established that in every integral class $M(L)(\mathbb{I}^2)$ with $\lim_{t\to\infty}\frac{M(t)}{t\ln t}=0$ there exists a function *f* for which $\overline{\lim_{n\to\infty}}\left|c_n(f,H^2)h_n(x)\right|=\infty$ on the set of positive measure.

The following generalization of Theorems A–D is true.

Theorem. *Let*

- $B_1,\dots,B_d\subset L^0(\mathbb{I})$ *be Banach spaces satisfying the* (*)*-condition;*
- $\Phi^{(1)},\dots,\Phi^{(d)}$ *be bases in* $B_1,\dots,B_d$, *respectively, and* $\left(\varphi_n\right)_{n\in\mathbb{N}^d}$ *be their product;*
- $\Psi^{(1)},\dots,\Psi^{(d)}$ *be systems of functions dual to* $\Phi^{(1)},\dots,\Phi^{(d)}$, *respectively;*
- $(n_k^{(1)}),\dots,(n_k^{(d)})$ *be sequences of indices corresponding to* $\Phi^{(1)},\dots,\Phi^{(d)}$, *respectively, according to Lemma A;*
- $M(L)(\mathbb{I}^d)$ *be an Orlicz space satisfying the* Δ'*-condition;*

- *spaces* $B_1,\ldots,B_d$ *be continuously embedded in* $M(L)(\mathbb{I}^d)$ *;*
- $\Psi^{(1)},\ldots,\Psi^{(d)} \subset M^*(L)(\mathbb{I}^d)$, *where* $M^*(L)$ *is the space dual to* $M(L)$.

Then for every $f \in M(L)\setminus L(\ln^+ L)^{d-1}(\mathbb{I}^d)$ *there exists an equimeasurable with f function g on* $\mathbb{I}^d$ *such that*

$$\overline{\lim_{k\to\infty}}\left|\sum_{m\in\Gamma(k)} c_m(g,\Phi^{(1)}\times\cdots\times\Phi^{(d)})\,\varphi_m(x)\right| = \infty$$

for almost every $x \in \mathbb{I}^d$, *where the blocks* $\Gamma(k)$ *are defined as follows*

$$\Gamma(k) = \left(\left[n_{k_1}^{(1)}+1, n_{k_1+1}^{(1)}\right]\times\cdots\times\left[n_{k_d}^{(d)}+1, n_{k_d+1}^{(d)}\right]\right)\cap\mathbb{N}^d \qquad (k\in\mathbb{N}^d).$$

Consequently, the Fourier series of g with respect to $\Phi^{(1)}\times\cdots\times\Phi^{(d)}$ *diverges almost everywhere.*

Remark. Obviously, if $M(L)(\mathbb{I}^d) = L(\mathbb{I}^d)$, then the conclusion of Theorem valid for any $f \in L\setminus L(\ln^+ L)^{d-1}(\mathbb{I}^d)$. In particular, this is the case if $\Phi^{(1)},\ldots,\Phi^{(d)}$ are complete orthonormal systems such that $\Phi^{(1)},\ldots,\Phi^{(d)} \subset L^\infty(\mathbb{I})$.

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