საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 7, №1, 2013
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 7, no. 1, 2013

Informatics

An Algorithm of the Solution of an Optimal Control Problem for a Nonlocal Problem

David Devadze*, Vakhtang Beridze*

* *Faculty of Education and Sciences, Shota Rustaveli State University, Batumi*

(Presented by Academy Member Archil Prangishvili)

ABSTRACT. The paper deals with optimal control problems whose behavior is described by Helmholtz equations with Bitsadze-Samarski nonlocal boundary conditions. A theorem about a necessary and sufficient condition of optimality is given. A numerical algorithm of the solution of an optimal control problem by means of the Mathcad package is presented. *© 2013 Bull. Georg. Natl. Acad. Sci.*

Key words: *Helmholtz equation, nonlocal boundary value problem, Bitsadze-Samarski problem, optimal control, optimality condition, Mathcad.*

Nonlocal boundary value problems are a very interesting generalization of classical problems and at the same time they are obtained in a natural manner when constructing mathematical models of real processes and phenomena in physics, engineering, sociology, ecology and so on [1-3]. The Bitsadze-Samarski nonlocal boundary value problem [4] arose in connection with mathematical modeling of processes occurring in plasma physics. Intensive studies of Bitsadze-Samarski nonlocal problems [5] and its various generalizations began in the 80s of the last century [4-8].

The present paper deals with optimal control problems whose behavior is described by Helmholtz equations with Bitsadze-Samarski nonlocal boundary conditions. Necessary optimality conditions are established by using the approach worked out in [9,10] for controlled systems of general type. Nonlocal boundary value problems and conjugate problems are solved by using the algorithm of reducing nonlocal boundary value problems to a sequence of Dirichlet problems [5].

In the paper the Bitsadze-Samarski boundary value problem for Helmholtz equations is considered. An optimal control problem is stated for a nonlocal boundary value problem with an integral quality test. A theorem about the necessary and sufficient optimality condition and a theorem about the existence and uniqueness of a solution of the adjoint problem are proved. A numerical algorithm of the solution of an optimal control problem by means of the Mathcad package is presented.

1. Statement of an Optimal Control Problem

Let the domain $\overline{G}$ be rectangular, $\overline{G}=[0,1]\times[0,1]$, Γ be the boundary of the domain G, $0<x_0<1$, $\gamma_0=\{(x_0,y):\ 0\le y\le 1\}$, $\gamma=\{(1,y):\ 0\le y\le 1\}$, $a(x,y),b(x,y),c(x,y),d(x,y)\in L_p(\overline{G})$, $p>2$, $0\le q(x,y)\in L_\infty(\overline{G})$, $U=[-1,1]$, Ω be the set of all admissible control functions $\omega(x,y):G\to U$.

For each fixed $\omega\in\Omega$ in the domain $\overline{G}$, let us consider the Bitsadze-Samarski boundary value problem for Helmholtz equations

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-q(x,y)u=a(x,y)\omega(x,y)+b(x,y),\ \ (x,y)\in G, \tag{1.1}$$
$$u(x,y)=0,\quad (x,y)\in\Gamma\setminus\gamma,$$
$$u(1,y)=\sigma u(x_0,y),\ \ 0\le y\le 1,\ \ \sigma>0.$$

Consider the function

$$I(\omega)=\iint_G\left[c(x,y)u(x,y)+d(x,y)\omega(x,y)\right]dx\,dy \tag{1.2}$$

and state the following optimal control problem: Find a function $\omega_0(x,y)\in\Omega$, for which the solution of problem (1.1) gives a minimal value to functional (1.2). A function $\omega_0(x,y)\in\Omega$ will be called an optimal control, and the corresponding solution $u_0(x,y)$ an optimal solution.

Theorem 1. *Let ψ_0 be a solution of the adjoint problem*

$$\begin{gathered}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-q(x,y)\psi=-c(x,y),\ \ (x,y)\in G\setminus\gamma_0,\\ \psi(x,y)=0,\ (x,y)\in\Gamma,\\ \frac{\partial\psi(x_0^+,y)}{\partial x}-\frac{\partial\psi(x_0^-,y)}{\partial x}=\sigma\frac{\partial\psi(1,y)}{\partial x},\ \ 0\le y\le 1,\end{gathered} \tag{1.3}$$

then for (u_0,ω_0) to be optimal it is necessary and sufficient that the principle of minimum

$$\inf_{\omega\in U}\left[d(x,y)+a(x,y)\psi_0(x,y)\right]\omega=\left[d(x,y)+a(x,y)\psi_0(x,y)\right]\omega_0 \tag{1.4}$$

be fulfilled almost everywhere on G.

2. Algorithm of the Solution of an Optimal Control Problem

The scheme of the solution of an optimal control problem is as follows:

- to find a solution $\psi_0(x,y)$, we first solve the adjoint problem (1.3);
- using the function $\psi_0(x,y)$ from (1.4), we construct the optimal control $\omega_0(x,y)$;
- to find an optimal solution $u_0(x,y)$, we solve problem (1.1).

A solution of the conjugate problem (1.3) is written in the form $\psi=w+w^*$, where w^* is the solution of the Dirichlet problem

$$\frac{\partial^2 w^*}{\partial x^2}+\frac{\partial^2 w^*}{\partial y^2}-q(x,y)w^*=-c(x,y),\ \ (x,y)\in G,\qquad (2.1)$$
$$w^*(x,y)=0,\quad (x,y)\in\Gamma,$$

And w is the solution of the nonclassical boundary value problem

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}-q(x,y)w=0,\ \ (x,y)\in G\setminus\gamma_0,$$
$$w(x,y)=0,\ \ (x,y)\in\Gamma,$$
$$\frac{\partial w(x_0^+,y)}{\partial x}-\frac{\partial w(x_0^-,y)}{\partial x}=\sigma\frac{\partial w(1,y)}{\partial x}+\sigma\frac{\partial w^*(1,y)}{\partial x},\ \ 0\le y\le 1.\qquad (2.2)$$

As is known [11], problem (2.1) has a unique solution that belongs to the space $\overset{\circ}{W}{}_2^2(G)$. To solve problem (2.2) we consider the iteration process

$$\frac{\partial^2 w^{k+1}}{\partial x^2}+\frac{\partial^2 w^{k+1}}{\partial y^2}-q(x,y)w^{k+1}=0,\ \ (x,y)\in G\setminus\gamma_0,$$
$$w^{k+1}(x,y)=0,\ \ (x,y)\in\Gamma,$$
$$\frac{\partial w^{k+1}(x_0^+,y)}{\partial x}-\frac{\partial w^{k+1}(x_0^-,y)}{\partial x}=\sigma\frac{\partial w^k(1,y)}{\partial x}+\varphi(y),\ \ 0\le y\le 1,\ \ k=0,1,2,\ldots,\qquad (2.3)$$

where $\varphi(y)=\sigma\frac{\partial w^*(1,y)}{\partial x}$, $w^0(x,y)$ is the initial approximation which is assumed to be equal to zero.

Problem (2.3) is equivalent to the following problem [12, 13]

$$\frac{\partial^2 w^{k+1}}{\partial x^2}+\frac{\partial^2 w^{k+1}}{\partial y^2}-q(x,y)w^{k+1}=\delta(x_0-x)\left(\sigma\frac{\partial w^k(1,y)}{\partial x}+\varphi(y)\right),\ \ (x,y)\in G,\qquad (2.4)$$
$$w^{k+1}(x,y)=0,\ \ (x,y)\in\Gamma,\ \ k=0,1,2,\ldots,$$

where $\delta(x_0-x)$ is the Dirac function.

Let us introduce the notation $\sigma_0=1/\left\|\frac{\partial G(1,y,x_0,\eta)}{\partial x}\right\|_{L_2([0,1]\times[0,1])}$, where $G(x,y,\xi,\eta)$ is the Green's function of problem (2.4).

Theorem 2. *Let* $0<\sigma\le\sigma_0$, *then the solution of problem* (2.4) *exists, is unique and belongs to the space* $W_2^2(G\setminus\gamma_0)\cap\overset{\circ}{W}{}_2^1(G)$.

To solve problem (1.1) we consider the iteration process

$$\frac{\partial^2 u^{k+1}}{\partial x^2}+\frac{\partial^2 u^{k+1}}{\partial y^2}-q(x,y)u^{k+1}=a(x,y)\omega+b(x,y),\ \ (x,y)\in G, \tag{2.5}$$

$$u^{k+1}(x,y)=0,\quad (x,y)\in\Gamma\setminus\gamma,$$

$$u^{k+1}(1,y)=\sigma u^{k}(x_0,y),\ \ 0\le y\le 1,\ \ \sigma>0,\ \ k=0,1,2,\ldots .$$

3. Numerical Realization of the Problem by Means of the Mathcad

To obtain a numerical solution of problems (2.4) and (2.5), at each iteration step we use the built-in function **relax(a, b, c, d, e, f, u, rjac)** in the Mathcad [14, 15].

The function **relax** returns the square matrix, where the position of an element in the matrix corresponds to its position inside the square domain, while the value corresponds to an approximate solution at this point.

The arguments of the function **relax** are as follows:

a, **b**, **c**, **d**, **e** are square matrices of one and the same size, containing the coefficients of a differential equation. In particular, for a Helmholtz equation the coefficients are $a_{i,j}=b_{i,j}=c_{i,j}=d_{i,j}=1,\ e_{i,j}=-4-q_{i,j}$, where $q_{i,j}$ are the values of $q(x,y)$ in the respective node inside the square domain;

f is a square matrix containing the values of the right-hand part of the equation in the respective node inside the square domain;

u is a square matrix containing the boundary values of the function at the domain edges, and also the initial approximation of the solution in the interior nodes of the square domain;

rjac is the parameter controlling the relaxation process convergence. It may vary in the range from 0 to 1, but an optimal value depends on the particulars of the problem.

As an example let us consider problems (2.1) and (2.5), where $q(x,y)=x+y$, $a(x,y)=1$, $b(x,y)=0$, $c(x,y)=1$, $d(x,y)=1$, $x_0=1/2$.

The results of the numerical solution of problems (2.1) and (2.5) are graphically shown in Figures 1 and 2.

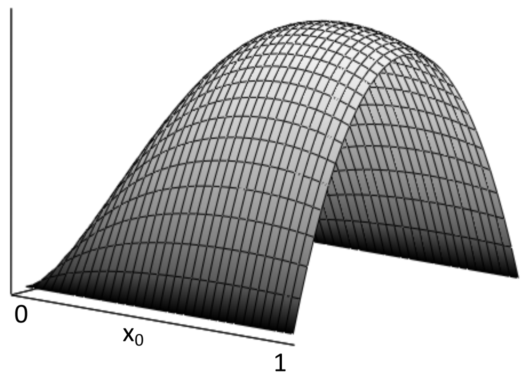

Fig. 1.

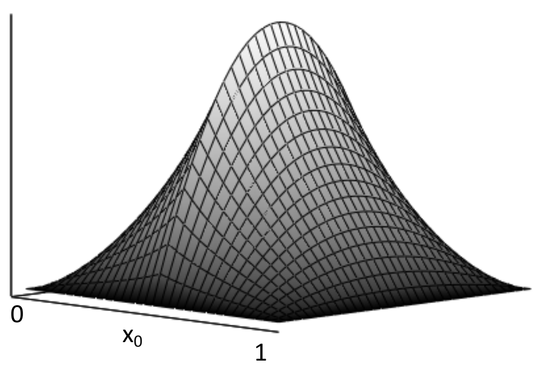

Fig. 2.

ინფორმატიკა

არალოკალური ამოცანისათვის ერთი ოპტიმალური მართვის ამოცანის ამოხსნის ალგორითმი

დ. დევაძე*, ვ. ბერიძე*

* *შოთა რუსთაველის სახელმწიფო უნივერსიტეტი განათლებისა და მეცნიერებათა ფაკულტეტი, ბათუმი*

(წარმოდგენილია აკადემიის წევრის ა. ფრანგიშვილის მიერ)

წინამდებარე ნაშრომში ჰელმჰოლცის განტოლებისათვის განხილულია ბიწაძე-სამარსკის არალოკალური სასაზღვრო ამოცანა, რომლისთვისაც დასმულია ოპტიმალური მართვის ამოცანა. ნაშრომში მოყვანილია თეორემები ოპტიმალობის აუცილებელი და საკმარისი პირობების და შეუღლებული განტოლების ამონახსნის არსებობისა და ერთადერთობის შესახებ. არალოკალური სასაზღვრო ამოცანის და შეუღლებული ამოცანის ამოხსნისათვის გამოყენებულია არალოკალური სასაზღვრო ამოცანის დირიხლეს ამოცანების მიმდევრობაზე დაყვანის ალგორითმი. წარმოდგენილია ოპტიმალური მართვის ამოცანის ამოხსნის ალგორითმი და რიცხვითი რეალიზაცია Mathcad-ის საშუალებით. რიცხვითი შედეგები წარმოდგენილია გრაფიკული სახით.

REFERENCES

1. *V.V. Shelukhin* (1993), Dinamika sploshn. sredy, No. 107: 180-193.
2. *C.V. Pao* (1995), J. Math. Anal. Appl. **195**, 3: 702-718.
3. *E.I. Obolashvili* (1992), Applicable Analysis, 45: 269–280. DOI:10.1080/00036819208840101
4. *A.V. Bitsadze*, *A.A. Samarskii* (1969), Dokl. Akad. Nauk SSSR **185**, 739-740 (in Russian); English transl.: (1969), Sov. Math., Dokl. **10**, 398-400.
5. *D.G. Gordeziani* (1986), O metodakh resheniia odnogo klassa nelokal'nyx kraevykh zadach. Tbilisi, (in Russian).
6. *D.V. Kapanadze* (1987), Differentsial'nye uravneniia **23**, 3: 543-545, 552 (in Russian).
7. *D. Gordeziani*, *N. Gordeziani*, *G. Avalishvili* (1998), Bull. Georg. Acad. Sci. **157**, 3: 365-368.
8. *G. Berikelashvili* (2006), J.Inequal. Appl., Art. ID 93858 : 1-6, doi:10.1155/JIA/2006/93858.
9. *V.I. Plotnikov* (1972), Izv. AN SSSR. Ser. Mat. **36**: 652-679 (in Russian).
10. *G.V. Meladze* (1987), in: Sovremennye problemy matematicheskoi fiziki. Trudy vsesoiuznogo simpoziuma, Tbilisi, 22-25 apr. 1987, **1**: 305-311 (in Russian).
11. *G.V. Meladze*, *T.S. Tsutsunava*, *D.Sh. Devadze* (1987), An optimal control problem for quasi-linear differential equations of first order on the plane with nonlocal boundary conditions. Tbilisi State University Press, Tbilisi, 1987, deposited at the Georgian Institute of Technical and Scientific Information, 25.12.87, No. 372, A87, 61 p.
12. *O.A. Ladyzhenskaya*, *N.N. Ural'tseva* (1973), Lineinye i kvazilineinye uravneniia elipticheskogo tipa. M., 576 p. (in Russian).
13. *V.S. Vladimirov* (1981), Uravneniia matematicheskoi fiziki. M. (in Russian).
14. *S.L. Sobolev* (1950), Uravneniia matematicheskoi fiziki. M.-L. (in Russian).
15. *D.V. Kiryanov* (2007), Mathcad 14. BHV-Petersburg, St. Petersburg.
16. *M. Herhager*, *J. Partoll* (2000), Mathcad 2000: The Complete Guide: Trans. with Ger.-K. Publishing Group, BHV.

Received October, 2012