Mechanics

Comparison of Analytical and Numerical Methods for Assessment of Stress-Strain State of the Massif around the Tunnel of Noncircular Cross-Section

Levan Japaridze

Academy Member, G.Tsulukidze Mining Institute. Tbilisi.

ABSTRACT. In the recent years in the world practice of calculation of the underground structures numerical methods have almost displaced the analytical methods of continuum mechanics. While numerical methods are indispensable for some intricate problems of geomechanics, the analytical methods of the theory of elasticity for calculation of tunnels should be a subject of further use and development. Parallel solution of specific problems with the use of one of the commercial computer software and N.Muskhelishvili's method of Theory of Elasticity was conducted. Using the example of tunnel of rectangular cross-section, it was shown that the analytical solution can be more accurate and comparatively easier over numeric especially using the well known program "Mathlab". *© 2013 Bull. Georg. Natl. Acad. Sci.*

Key words: *underground structures, calculation, analytical methods, numerical methods.*

In the last 30s-40s, together with the development of computer technologies numerical methods of the mechanics of continua have evolved rapidly and effectively. Thus it is frequently noted [1,2 etc.] that the closed form solution is restricted to simple geometries and material models, and therefore are often of limited practical value and the solution is considered to be a good tool for the assessment of the results obtained from numerical analysis.

The market rate of computer programs for this purpose is increasing. To solve problems of underground structures some international companies have also designed rather expensive special commercial programs ("Rock science", "Phase 2. 7", "Examine", "Flac", etc.). They are most effective for modeling the so-called multiply-connected domains which occur, for example, in processing minerals when underground excavations of different forms of cross-section are located close to each other.

Analytical methods of the elasticity theory are often forgotten even for less complex form of underground structures, for which they are quite helpful and often more accurate than numerical methods. Today in the world practice of calculation of tunnels use is often made of this well-known analytical solution of the task of stress-deformed state around a circular hole, made in 1898 by the German engineer Kirsch (1898).

The most powerful technique for finding the stresses and displacements around two-dimensional

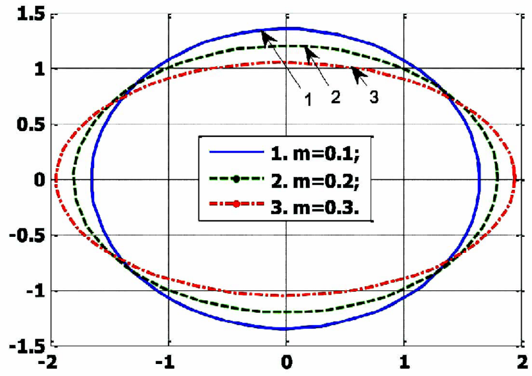

Fig. 1. Family of curves, mapping of elliptical cross-section of tunnel.

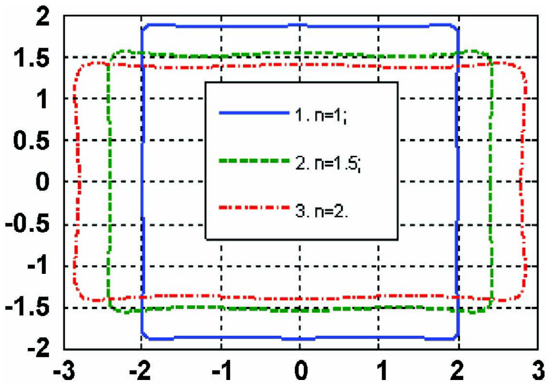

Fig. 2. Family of curves, mapping of rectangular cross-section of tunnel.

holes - methods of the theory of elasticity, developed by N. Muskhelishvili [2], is used relatively rarely. Meantime the opportunity of this theory for modeling of practically any form of tunnel cross-section is recognized by a number of authoritative scientists [1,3,4 etc.]. It is considered that solution of such tasks by analytical methods is difficult, because full exploitation of this method requires knowledge of the various integral theorems of complex analysis [1]. If so, in order to facilitate the analytical solution the same computer technologies, in particular the program "Mathlab" could be used. This will greatly simplify and make clear practical calculations with high accuracy.

Radial σ_r and tangential σ_θ stress and radial v_r and tangential v_θ displacements for a single hole of any form in an infinite isotropic elastic medium under plane strain conditions, are defined by N.Muskhelishvili's [2] known equations:

$$\sigma_r + \sigma_\theta = 4\,\mathrm{Re}\frac{\varphi'}{\omega'(\zeta)}; \quad (1)$$

$$\sigma_r - \sigma_\theta + 2i\tau_{r\theta} = \frac{2\zeta^2}{\rho^2\overline{\omega}'(\overline{\zeta})}\times$$

$$\times\left[\overline{\omega}(\overline{\zeta})\frac{\varphi''(\zeta)\omega'(\zeta)-\varphi'(\zeta)\omega''(\zeta)}{\omega'(\zeta)^2}-\frac{\psi'(\zeta)}{\omega'(\zeta)}\right]; (2)$$

$$2\mu(v_r + iv_\theta) = \frac{\overline{\zeta}}{\rho}\frac{\overline{\omega}'(\overline{\zeta})}{|\omega'(\zeta)|}$$

$$\times\left[\kappa\varphi(\zeta)-\frac{\omega(\zeta)}{\overline{\omega}'(\zeta)}\overline{\varphi}'(\zeta)-\overline{\psi}(\zeta)\right]; \quad (3)$$

$$\mu = E/2(1+\nu) \quad \kappa = 3-4\nu;$$

where: E and ν are Young's module and Poisson's ratio of the massif.

$\varphi(\zeta)$ and $\psi(\zeta)$ - analytical functions of the complex variables $\zeta = re^{i\vartheta}$, that depend on boundary conditions;

$\omega(\zeta)$ - function of conformal mapping for given form of opening's contour.

Schwarz–Christoffel function of conformal mapping for contour of the single hole looks like:

$$\omega(\zeta) = iR\left(A\zeta^{-1} + B\zeta + C\zeta^2 + D\zeta^3 + E\zeta^5 + \ldots\right), (4)$$

Values of the coefficients: R, A, B, C, D, E ... designate the shape of the contour. For example, when C=D=E=0, B=1 and A=(a-b)/(a+b), function (4) designates an ellipse with semiaxis a and b (Fig.1).

When mapping contour is of rectangular shape with width - l and height –h (Fig.2), the coefficients of (4) will look like:

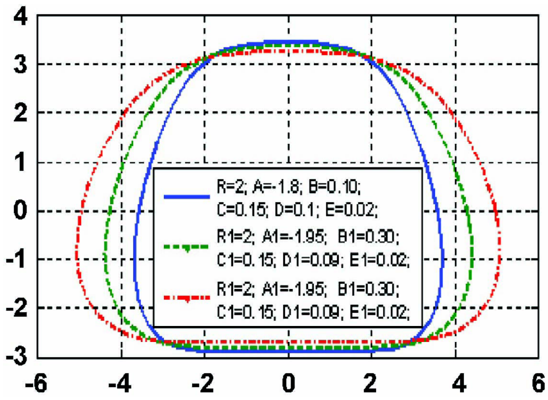

Fig. 3. Family of curves, mapping of arched cross-section of tunnel.

$$A=1;\ B=\frac{(a+\bar{a})}{2};\ C=0;\ D=\frac{(a-\bar{a})^2}{24};$$

$$E=\frac{(a^2-\bar{a}^2)(a-\bar{a})}{80};\quad R=\frac{1}{(1+B+C+D)};$$

where: $a=e^{2k\pi i}$; $\bar{a}=e^{-2k\pi i}$; $k=0.25(n)^{-0.3}$; $n=l/h$.

Fig. 3 presents the given family of curves, mapping of arched cross-section of tunnels. For this case the above-mentioned coefficients are: R=2; A=-1.8; B=0.10; C=0.15; D=0.1; E=0.02; with their small changes to get the needed form using the "Mathlab".

As an example, consider the problem of equilibrium of unevenly loaded infinite homogeneous, isotropic plane with nearly rectangular hole (Fig.4) using analytical method and computer program "Mathlab".

The results will be compared with the numerical solution of using the special program "phase 2.7", "Rocksciense".

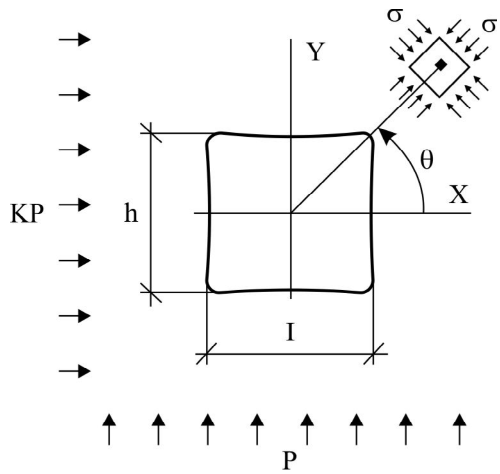

Fig. 4. Nearly rectangular hole l/h=1, subjected to a far-field vertical - P=10MPa and horizontal - K*P=5 MPa stress.

For a given base schema coefficients of conformal mapping function (4) are: R=A=1; B=C=E=0; D=1/6, and appropriate complex stress potentials [1] can be written so:

$$\varphi(\zeta)=PR\big[(1+K)\times$$
$$\times\left(0.25\zeta^{-1}+0.0415\zeta^{3}\right)-0.428(1-K)\zeta^{3}\big] \quad (5)$$

$$\psi(\zeta)=-PR\left[(1+K)\left(\frac{1}{2\zeta}+\frac{1.083}{2+\zeta^{4}}\right)-\right.$$
$$\left.-(1-K)\frac{0.925\zeta^{3}}{\left(2+\zeta^{2}\right)}\right] \quad (6)$$

Substituting these meanings of functions (5), (6), their derivatives and conjugates in the equations (1)-(3), separating the real and imaginary parts, we

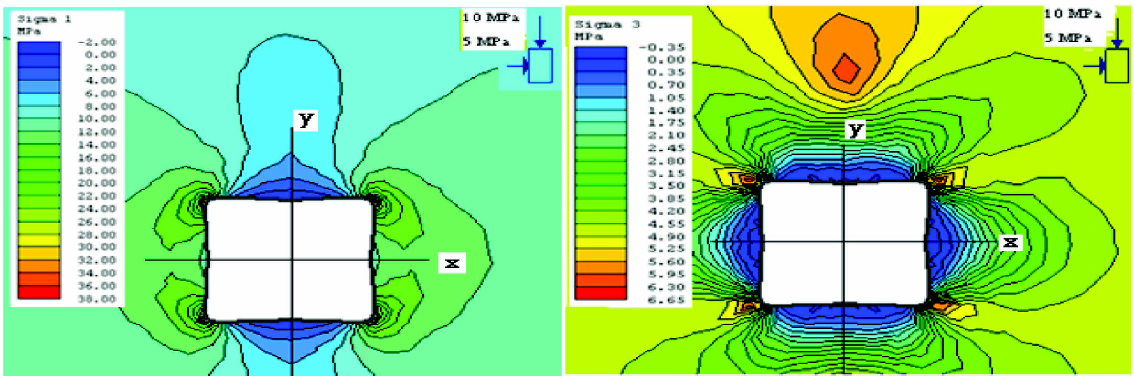

Fig.5. Fields of the XX and YY elastic stress around the tunnel rectangular cross-section according to "Rockscience", Phase2. 7".

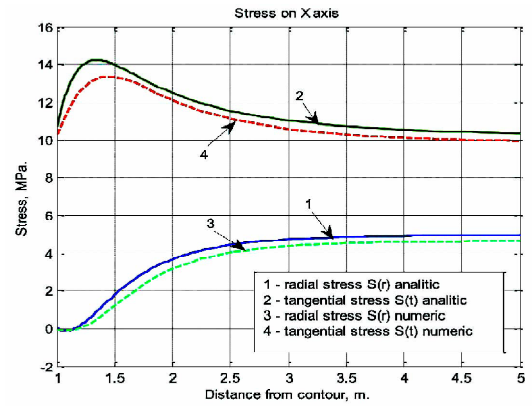

Fig. 6. Distribution of radial and tangential stress on X axis from contour to 5m, calculated using analytical and numerical methods.

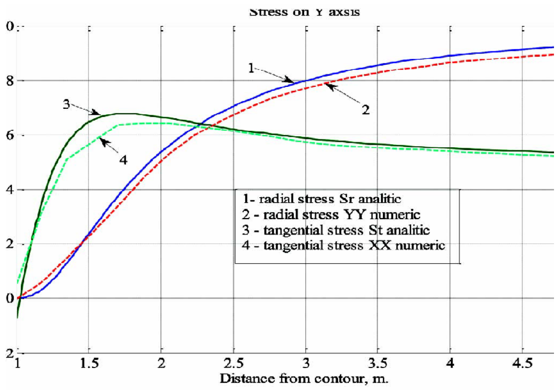

Fig. 7. Distribution of radial and tangential stress on Y axis from contour to 5m, calculated using analytical and numerical methods.

shall receive components of stress and/or displacements in any point of plane. With this routine task we will easily execute "Mathlab", an exemplary program of which for our task is given in Table.

Table. "Mathlab" exemplary program for calculation of stress around the tunnel rectangular cross-section

```
1.teta=pi/2; ro=0.2:0.01:1; x=1./ro; R=1; z=ro*exp(i*teta);
2.om = R*(1./z-z.^3/6); omm= R*(-1./z.^2-z.^2/2);
3.ommm=R*(2./z.^3-z);
4.omsh = R*(1./ro*exp(i*teta)-0.1666*ro.^3*exp(-3*i*teta));
5.ommsh= R*(-1./ro.^2*exp(2*i*teta)-0.5*ro.^2*exp(-2*i*teta));
6.alfa=pi/2;  P=10; T=(3*cos(2*alfa)/7+i*3*sin(2*alfa)/5);
7.fiP=P*R/4.*(1./z-0.166.*z.^3)+P*R*(T.*z+z.^3/12);
8.fiiP=P*R/4.*(-1./z.^2-0.5.*z.^2)+P*R.*(T+z.^2/4);
9.fiiiP=P*R/4.*(2./z.^3-z)+P*R/2.*z;
10.ksiP=-P*R/2.*(1./z-0.166.*z.^3).*exp(-2*i*alfa)-
     P*R/12.*(z.^3*exp(2*i*alfa)+(13.*z-26*T.*z.^3)./(2+z.^4));
11.ksiiP=-P*R/2.*(-1./z.^2-0.5.*z.^2).*exp(-2*i*alfa)-
      P*R/12.*(3*z.^2*exp(2*i*alfa)+((13-78*T.*z.^2).*(2+z.^4)-
     (13.*z-26*T.*z.^3).*4.*z.^3)./(2+z.^4).^2);
12.alfa1=0;  K=5;  T1=(3*cos(2*alfa1)/7+i*3*sin(2*alfa1)/5);
13.fiK=K*R/4.*(1./z-0.166.*z.^3)+K*R*(T1.*z+z.^3/12);
14.fiiK=K*R/4.*(-1./z.^2-0.5.*z.^2)+K*R.*(T1+z.^2/4);
15.fiiiK=K*R/4.*(2./z.^3-z)+K*R/2.*z;
16.ksiK=-K*R/2.*(1./z-0.166.*z.^3).*exp(-2*i*alfa1)-
     K*R/12.*(z.^3*exp(2*i*alfa1)+(13.*z-26*T1.*z.^3)./(2+z.^4));
17.ksiiK=-K*R/2.*(-1./z.^2-0.5.*z.^2).*exp(-2*i*alfa1)-
     K*R/12.*(3*z.^2*exp(2*i*alfa1)+((13-78*T1.*z.^2).*(2+z.^4)-
    (13.*z-26*T1.*z.^3).*4.*z.^3)./(2+z.^4).^2);
18.FIK=fiiK./omm; FIIK=(fiiiK.*omm-fiiK.*ommm)./omm.^2; 19.KSIK=ksiiK./omm;
20.FIP=fiiP./omm; FIIP=(fiiiP.*omm-fiiP.*ommm)./omm.^2; 21.KSIP=ksiiP./omm;
22.FI=FIP+FIK;  FII=FIIP+FIIK;  KSI=KSIP+KSIK;
23.M=2.*z.^2./(ro.^2.*ommsh);
24.A = 4*real(FI);    B = real(M.*(omsh.*FII+omm.*KSI));
25.Sr=(A-B)/2;  St=(A+B)/2;
26.r=1+[0 0.17 0.35 0.52 0.70 … 3.16 3.34 3.51 3.69 3.86 4.00];
27.XX=[ 0.5605 3.18 5.10 5.75 … 5.35 5.31 5.27 5.24 5.22 5.19];
28.YY=[-0.0065 0.60 1.48 2.42 … 8.72 8.79 8.87 8.94 9.01 9.04];
29.plot(x,Sr,x,St,r,XX,r,YY);  grid on
```

Positions 27-29 of Table 1 represent the result of numeric solution of the same task with the same boundary conditions, solved by a numerical method using the program "Phase 2.7". They are imported from Fig. 5 using "Excel" embedded in the "Phase2.7".

So, "Mathlab" gives the superposed graphics of analytic and numerical solution of the problem (Fig. 6, Fig. 7):

Comparative analyses of the superposed graphics of analytic and numerical solution of the problem (Fig. 6, Fig. 7) allow us to make the following conclusions:

1. Comparisons of calculation results of the stress components around tunnels, obtained using: analytical method based on the functions of complex variable, and computer program for geotechnic objects ("Rockscience", Phase 2.7), based on finite elements method, significantly differ from each other.

2. If in the case of axisymmetric problems, considered in "Phase 2.7 Stress verification manual", error can be no more than 2-3%, then at the non-round holes and irregular "in-situ far-field" stress, such precision can take place at only some points, for example on the X axis (Fig. 6). At other places, for example on the Y axis (Fig. 7), the quantative error is bigger.

3. Powerful methods of the theory of elasticity, developed by N.Muskhelishvili for the solution of problems of elastic stress-deformed state of massif around tunnels of practically any form of cross-section, make possible the solution much more precisely and easily, especially using the computer program "Mathlab".

მექანიკა

არაწრიული კვეთის გვირაბის ირგვლივ ქანების მასივის დაძაბულ-დეფორმირებული მდგომარეობის შეფასებისათვის ანალიზური და რიცხვითი მეთოდების შედარება

ლ. ჯაფარიძე

აკადემიის წევრი, გ.წულუკიძის სამთო ინსტიტუტი. თბილისი.

ბოლო 30-40 წლის განმავლობაში მიწისქვეშა ნაგებობების გაანგარიშების მსოფლიო პრაქტიკაში უწყვეტი ტანის მექანიკის ანალიზური მეთოდები ფაქტიურად განდევნა რიცხვითმა მეთოდებმა. თუ ეს უკანასკნელი შეუცვლელია გეომექანიკის რთული სამგანზომილებიანი ამოცანებისათვის, ბრტყელი ამოცანების შემთხვევებში დრეკადობის თეორიის ანალიზური მეთოდები კვლავაც უნდა დარჩეს გამოყენებისა და განვითარების საგნად. ჩატარებულია მიწისქვეშა ნაგებობების კერძო ამოცანების ამოხსნა. გამოყენებულია საერთაშორისო მასშტაბით გავრცელებული სასრულ ელემენტთა მეთოდზე დამყარებული სპეციალური კომპიუტერული პროგრამა და ნ.მუსხელიშვილის

დრეკადობის თეორიის ანალიზური აპარატი. განივკვეთის მართკუთხედთან მიახლოებული ფორმის გვირაბის მაგალითზე მიღებული შედეგების შედარებითი ანალიზით ნაჩვენებია, რომ ანალიზური ამონახსნი უფრო ზუსტია ამასთან ადვილიც, თუ პარალელურად გამოყენებული იქნება კომპიუტერული პროგრამა "მათლაბი".

REFERENCES

1. *J.C. Jaeger, N.C.W. Cook*, and *R.W. Zimmerman,* (2007), Fundamentals of Rock Mechanics. Blackwell Publishing, 255 p.
2. *N.I. Muskhelishvili,* (1963), Some Basic Problems of the Mathematical Theory of Elasticity, 4th ed.,Noordhoff,Groningen.
3. *K.V. Ruppeneit*, *I.M. Liberman* (1960), Vvedenie v mekhaniku gornykh porod. Moscow, 356 p. (in Russian).
4. *G. N. Savin* (1961), Stress Concentration Around Holes, New York.
5. *L.A. Japaridze* (1991), Osnovnye voprosy rascheta po predel'nym sostoianiiam krepi podzemnykh sooruzhenii. Moscow (in Russian)
6. Stress Analysis Verification Manual. Version 7.0. 1989-2008 Rockscience Inc.

Received December, 2012