საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 7, №1, 2013
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 7, no. 1, 2013

Mathematics

On Poncelet Porism for Biquadratic Curves

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ABSTRACT. We discuss two seemingly unrelated topics having in fact a common feature that they naturally lead to consideration of certain involutive transformations of biquadratic curves. The first topic is concerned with the so-called Darboux transformation on the moduli space of planar quadrilateral linkage. We explain how this transformation can be related to involutions of an appropriate biquadratic curve and present a natural analog of Poncelet porism in this setting. The second topic is concerned with the uniqueness of solution to the Dirichlet problem for string equation in bounded domain. If the boundary is a convex biquadratic curve we show that an analog of Poncelet porism for the so-called John's mapping can be established in the same way as for Darboux transformation.

Key words: *biquadratic curve, product of involutions, Poncelet porism, quadrilateral linkage, moduli space, diagonal involution, Darboux transformation, Euler four points relation, elliptic curve, Dirichlet problem for string equation, John's mapping, bicentric polygon, poristic polygons.*

1. An interesting version of the famous *Poncelet porism* (big Poncelet theorem) [1] has recently been obtained by J.Duistermaat in the context of biquadratic curves [2]. The general approach developed by J.Duistermaat has a number of concrete implementations and applications, part of which can be found in [2]. In this note we discuss further aspects of the aforementioned version of Poncelet porism concerned with two seemingly unrelated topics. The unifying element of our considerations is a certain transformation of biquadratic curve naturally arising in each setting.

The first topic deals with the so-called *Darboux transformation* of moduli space of planar quadrilateral linkage [2, 3], while the second one is concerned with the uniqueness of solution to the Dirichlet problem for string equation in bounded domain investigated in [4-6]. Specifically, we obtain an analog of Poncelet porism for quadrilateral linkages and show that, for a domain bounded by convex biquadratic curve, certain results of [4] can be derived from the results of [2]. The uniqueness problem for string equation in domain bounded by biquadratic curve has been discussed in [6] but without referring to [2]. We believe that the results of [2] yield a unifying approach to both these topics which may suggest further developments concerned with biquadratic curves.

To provide a general background for our considerations let us recall that biquadratic curve X is defined by a polynomial of the form $P(x,y) = \Sigma\, a_{mn}\, x^m y^n$, where the indices m, n in each monomial can only take values 0, 1, 2. Recall also that a closed curve X is called convex with respect to (x,y)-directions if each horizontal or vertical line L which intersects X, either has two intersection points with X or has one tangency point with X.

Given a closed biquadratic curve with this property one can define two involutions of X called the horizontal switch S_1 and vertical switch S_2 [2]. For each point p=(x,y) of X, consider the horizontal line passing through p. By our condition it either touches X at p or has another intersection point $p' = (x, y')$. In the first case we put $S_1(p) = p$ while in the second case we put $S_1(p) = p'$. The vertical switch S_2 is defined analogously using vertical lines. It is obvious that both these mappings are involutions. Following [2] we also introduce the so-called *QRT-mapping* on X defined as $T = S_1 S_2$. As is explained in [2], this mapping plays an important role in many topics related to discrete integrable systems.

The mapping T is the main object of interest in the sequel. Specifically, we are interested in studying its iterations T^n and periodic points. As was shown in [2], in this setting one has an analog of Poncelet porism for QRT-mapping. Namely, the following dichotomy holds: either T has no periodic points in X or each point of X is periodic with the same period [2]. We are now aiming at discussing consequences of this general result in the settings mentioned above.

2. In order to describe the first setting we need some definitions and results concerned with polygonal linkages. Recall that a polygonal linkage (or a closed polygonal k-chain) L is defined by a k-tuple of positive numbers l_i called *sidelengths* of L. In the case of a closed polygonal chain it is always assumed that each of the sidelengths is not greater than the sum of all other ones [7]. The *planar configuration space* C(L) of a polygonal k-chain L is defined as the collection of all k-tuples of points v_i in Euclidean plane such that the distance between v_i and v_{i+1} is equal to l_i, where it is assumed that $v_{k+1} = v_1$. Each such collection of points is called a *configuration* of L. Factoring C(L) over the natural diagonal action of the orientation preserving isometries one obtains the *(planar) moduli space* $M(L) = M_2(L)$ [7]. Moduli spaces, as well as configuration spaces, are endowed with natural topologies induced by Euclidean metric.

It is obvious that the planar moduli space can be identified with the subset of configurations such that $v_1 = (0,0)$, $v_2 = (l_1,0)$. It is also easy to see that, for a closed k-chain, the moduli space has a natural structure of compact orientable real-algebraic set of dimension $k - 3$. Let us say that a polygonal linkage is *degenerate* if it has an *aligned configuration*, i.e., a configuration where all vertices lie on the same straight line. It is well-known that this happens if and only if there exists a k-tuple of "signs" $s_i = \pm 1$ such that $\Sigma\, s_i l_i = 0$. The moduli space M(L) of polygonal linkage L is smooth (does not have singular points) if and only if L is nondegenerate (see, e.g., [7]).

In the sequel we only consider *quadrilateral (4-bar) linkages*. So let Q = Q(a, b, c, d) be a nondegenerate quadrilateral linkage with pairwise non-equal lengths of the sides. For brevity, such a linkage will be called *admissible*. For each configuration V of admissible linkage Q, both diagonals have non-vanishing length so one can define the reflections $R_1(V)$ and $R_2(V)$ of V in the diagonal v_1v_3 and v_2v_4 respectively. In this way we obtain two involutions R_1, R_2 of the moduli space M(Q). Their composition $T = R_1R_2$ is called the *Darboux transformation* of linkage Q [2].

A natural problem is to investigate the discrete dynamical system $\{T^n\}$ on M(Q) generated by T. The approach of [2] enables one to obtain comprehensive results about the qualitative behaviour of the discrete dynamical system $\{T^n\}$ generated by T. One of the most spectacular results in this direction is an analog of

Poncelet porism for a general biquadratic curve. As is mentioned on page 512 of [2], this result in the context of quadrilateral linkages was discussed in a colloquium talk of the present author in Utrecht on 9.02.2006, which gave an impetus for the study initiated by J.Duistermaat.

Later on, it turned out that a version of Poncelet porism for quadrilateral linkages had already been known to G.Darboux. For this reason J.Duistermaat suggested to call T the *Darboux transformation* of a 4-bar linkage Q. According to [2] the original proof of this result by G.Darboux used the theory of elliptic functions and was rather involved. Below we present an outline of a simplified proof based on the approach of [2].

Along with M(Q) we will refer to its complex projectivization $M_C(Q)$ defined in a standard way by considering the equation of M(Q) in homogeneous coordinates [2]. It is known that both M(Q) and $M_C(Q)$ are smooth (do not have singular points) if and only if the sidelengths satisfy the aforementioned non-degeneracy condition. This condition, traditionally called the *Grashof condition*, actually means that Q does not have *aligned* configurations, or, equivalently, there do not exist numbers $s_i = \pm 1$ such that $\Sigma\, s_i l_i = 0$. Now we can formulate an analog of Poncelet porism for Darboux transformation of Q.

Theorem 1. *For an admissible 4-bar linkage Q, one has the following dichotomy: either each configuration is periodic with the same period or the orbit of each configuration is infinite.*

The idea of proof borrowed from [2] is quite elegant: one shows that T can be realized as an automorphism of the complexified planar moduli space $M_C(Q)$ acting as a translation, which makes the statement evident. The following outline of the proof contains all essential ingredients of the argument.

Outline of proof of Theorem 1. Let us use a rigid motion to place the first two vertices of Q=Q(a,b,c,d) at points $v_1 = (0,0)$, $v_2 = (a,0)$ and consider an angular parametrization of M(Q) by putting $v_3 = (a + b \cos s, b \sin s)$, $v_4 = (d \cos t, d \sin t)$. Then using the "tangent of half-angle" substitution, the remaining distance condition $d(v_3, v_4) = c$ can be rewritten in the form

$$((a+b+d)^2 - c^2)\, u^2 v^2 + ((a+b-d)^2 - c^2))u^2 + ((a-b+d)^2 - c^2)v^2 + (-a+b+d)^2 - c^2 = 0.$$

The above curve in the (u,v)-plane is biquadratic and, as explained in [2], if it is smooth as a curve in the Cartesian square of the complex projective line it is an elliptic curve. Since we assume that Q(l) satisfies the Grashof condition, this curve is smooth and $M_C(Q)$ is indeed an elliptic curve. It is now easy to directly verify that the Darboux transformation corresponds to the horizontal switch from (u,v) to (u′,v) followed by the vertical switch from (u′,v) to (u′,v′) in the (u,v)-plane. Therefore the Darboux transformation T on M(Q) coincides with the QRT-mapping of the above biquadratic curve. Then, as is shown in [2], the Darboux transformation acts on it as translation. Since for a translation it's obvious that it either has no periodic points or all points are periodic with the same period, the result follows.

From our viewpoint it is remarkable that this result can also be proved using another biquadratic curve naturally associated with linkage Q. This is achieved by taking the lengths of diagonals of configuration V as coordinates on M(L). More precisely, we put $x = d(v_1, v_3)$, $y = d(v_2, v_4)$ and notice that the pair (x,y) completely determines the shape of configuration V, i.e. its class in M(Q). Then an elementary classical result known as *Euler four points relation* [1] yields that the moduli space M(Q) is defined by the following equation in coordinates (x,y):

$$x^2y^4 + y^2x^4 - (a^2 + b^2 + c^2 + d^2)\, x^2y^2 + (a^2 - d^2)(b^2 - c^2)\, x^2 + (a^2 - b^2)(d^2 - c^2)\, y^2 + C_1 = 0,$$

where $C_1 = (b^2d^2 - a^2c^2)(b^2 + d^2 - a^2 - c^2)$ is a constant. Taking the squares x^2 and y^2 as new variables, we obtain another convex biquadratic curve. Since each diagonal involution does not change the length of one of the diagonals, they coincide with the horizontal and vertical shifts introduced above and so their composition coincides with the Darboux transformation. Now the proof can be completed in the same way as above.

Our considerations and results of [3-7] enable one to obtain similar results for open 3-bar linkages (3-arms) and spherical quadrilateral linkages. These generalizations will be discussed elsewhere.

3. Our second topic is concerned with the string equation $u_{xy} = 0$ considered in a bounded domain D. We wish to study the uniqueness of solutions to Dirichlet problem $\{u_{xy} = 0, u|C = f\}$, where f is a given continuous function on the boundary C = b(D). Recall that results of F.John [4] and N.Vakhania [5] revealed some interesting phenomena described in terms of the geometry of boundary C and its position with respect to characteristics of the string equation. Specifically, it appeared useful to investigate the periodic points of the so-called *characteristic billiard* in D [4] and several interesting results have been obtained in this way [4-5].

In particular, for a rectangular D, the uniqueness of solution to homogeneous Dirichlet problem takes place if and only if the ratio of the lengths of sides of D is irrational [4], [5]. Analogous results have been obtained for ellipses [4]. In a more general setting where it is only assumed that the boundary C is convex with respect to the family of characteristics of the string equation, F.John showed that the uniqueness problem is related to the periodic points of a certain transformation of C called nowadays the *John's mapping* $J: C \to C$. In particular, F.John formulated four alternatives in terms of the existence and cardinality of periodic points of J and showed their relation to the uniqueness problem.

Recently, V.Burskii and A.Zhedanov investigated the case where the boundary C is a convex biquadratic curve [6]. They showed that some of the alternatives suggested by F.John are impossible in this case and, generically, there holds the dichotomy: either mapping J has no periodic points or all points are periodic with the same period. In the first case one has uniqueness for the homogeneous Dirichlet problem while in the second case the uniqueness fails and one can suggest an explicit geometric construction of nontrivial solutions to the homogeneous Dirichlet problem [6]. Since in this case everything is determined by the dynamical properties of J, which is a purely geometric object, it is not surprising that the same conclusion can be derived using the general scheme of [2].

Namely, it is easy to verify that in this case the John's mapping coincides with the QRT-mapping of the corresponding biquadratic curve. Hence the dichotomy established in [6] becomes a direct consequence of Poncelet porism for QRT mapping established in [2]. Combining this with the known results on the correctness of the above Dirichlet problem we obtain the following criterion of uniqueness.

Theorem 2. *Let D be a domain bounded by a closed convex biquadratic curve C. Then the homogeneous Dirichlet problem* $\{u_{xy} = 0, u|C = 0\}$ *has only trivial solution in the space of continuously differentiable functions with square-integrable second derivatives if and only if the* QRT-*mapping of* C *has no periodic points.*

This criterion may be useful because the geometry of QRT-mapping is well-understood. In particular, a criterion of periodicity and an algorithm for computing the period of QRT-mapping can be found in [2]. The results in the case of elliptic boundary [4] become direct consequences of these general results. Moreover, the criterion of periodicity of QRT-mapping given in [2] enables one to effectively check the uniqueness property for any concrete biquadratic curve without assuming that the equation of C is given in a canonical form as is assumed for similar considerations in [6]. Concrete examples will be given in a detailed version of this paper.

4. We conclude by indicating a seemingly interesting research perspective suggested by the connections described in this note. A natural problem in the context of Poncelet porism is to describe the deformations of

a given biquadratic curve with periodic QRT-mapping which preserve the periodicity of QRT-mapping. Then such a deformation preserves also the order (length of period) of QRT-mapping and for this reason we call it *isoperiodic deformation* of biquadratic curve. An interesting example of description of isoperiodic deformations, which may serve as a paradigm for the general case, is given by *poristic bicentric polygons* [1].

Recall that a family of poristic bicentric polygons is defined by a pair of circles C, S such that C is inside S. Let us assume that there exists a bicentric n-gon associated with the pair (C, S), i.e. an n-gon inscribed in S and circumscribed about C. Then, by the big Poncelet theorem, there exists a one-dimensional family of bicentric n-gons accociated with (C,S). These n-gons are called poristic polygons associated with (C,S) and the pair (C,S) is called a poristic pair of circles of order n (the original German term is "Kreise in Schliessungslage" [1]). As is well-known, such a pair of circles defines a biquadratic curve X describing the Poncelet process [1] and it can be checked that the number n coincides with the period of QRT-mapping for X [2].

Up to a rigid motion such a situation is described by a triple of positive numbers (R, r, d), R > r > d, where R is the radius of external circle, r is the radius of internal circle, and d is the distance between the centers of these circles. It is known that in order that pair (C,S) be a poristic pair of order n, the numbers R, r, d should satisfy a certain algebraic relation. For example, for n=3, this is the Euler formula $R^2 - 2Rr = d^2$, and for n=4, this is the so-called Fuss relation

$$\frac{1}{(R-d)^2} + \frac{1}{(R+d)^2} = \frac{1}{r^2}.$$

For small values of n, it is known that these relations, called generalized Fuss relations, are in fact criteria, i.e. they guarantee that a given pair of circles is poristic of order n.

Notice that this gives a description of isoperiodic deformations of the biquadratic curve $X=X(R,r,d)$. Namely, it is now obvious that we obtain a two-dimensional family of isoperiodic deformations of X described by the condition that parameters R,r,d satisfy a generalized Fuss relation. It is not difficult to rewrite these relations in terms of the coefficients of biquadratic curve $X(R,r,d)$ and obtain an explicit description of its isoperiodic deformations. Examples of such descriptions will be given in a detailed version of this paper.

Now, by a way of analogy it would be interesting to find a similar description of isoperiodic deformations for a general biquadratic curve with periodic QRT-mapping and, in particular, find out what is the dimension of the set of isoperiodic deformations.

მათემატიკა

პონსელეს პორიზმი ბიკვადრატული წირებისთვის

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(წარმოდგენილია აკადემიკოს ნ. ვახანიას მიერ)

ნაშრომში განხილულია პონსელეს პორიზმი ბიკვადრატული წირებისთვის და მოყვანილია ამ პორიზმის ანალოგები სახსრული ოთხკუთხედის დარბუს გარდაქმნის შემთხვევაში და სიმის განტოლების მახასიათებელი ბილიარდის შემთხვევაში.

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Received January, 2012