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Mathematics

Lattice Isomorphisms of Nilpotent and Free Lie Algebras

Alexander Lashkhi

Department of Theoretical Informatics and Computer Modelling, Georgian Technical University, Tbilisi

(Presented by Academy Member Hvedri Inassaridze)

ABSTRACT. $\mathcal{K}$ **denotes a commutative domain with unit.** ***L*** **denotes the Lie algebra over** $\mathcal{K}$ **;** $\mathcal{L}(L)$ **is the lattice of all subalgebras** ***L*****. Let** $\mathcal{M}$ **and** $\mathcal{M}_1$ **be the linear algebras over the rings** $\mathcal{K}$ **and** $\mathcal{K}_1$**, respectively, and** $\sigma:\mathcal{K}\to\mathcal{K}_1$ **be an isomorphism. A bijection** $\mu:\mathcal{M}\to\mathcal{M}_1$ **will be called a** σ **-semilinear quasimorphism if for any** $x_1,x_2\in\mathcal{M}$ **and** $\alpha,\beta\in\mathcal{K}$ **there exists** $\lambda\in\mathcal{K}_1$ **such that**

$$\mu(\alpha x_1+\beta x_2)=\sigma(\alpha)\mu(x_1)+\sigma(\beta)\mu(x_2),\quad \mu(x_1x_2)=\lambda\mu(x_1)\mu(x_2).$$

Let $f:\mathcal{L}(L)\to\mathcal{L}(L_1)$ **be a lattice isomorphism, where** L **and** L_1 **are torsion-free nilpotent Lie algebras of class 2 over the principal ideal domains** $\mathcal{K}$ **and** $\mathcal{K}_1$**, respectively. If** $\dim L\neq 3$**, then** L **and** L_1 **are semilinear isomorphic.**

Let $f:\mathcal{L}(L)\to\mathcal{L}(G)$ **be a lattice isomorphism,** ***L*** **and** ***G*** **be Lie algebras over the principal ideal domains** $\mathcal{K}$ **and** $\mathcal{K}_1$**, respectively. If** ***L*** **is a free (nonabelian) polynilpotent or free Lie algebra and** $\dim L\neq 3$**, then** $\mathcal{K}\cong\mathcal{K}_1$ **and** $L\cong G$**.** *© 2013 Bull. Georg. Natl. Acad. Sci.*

Key words: *Lie algebra, lattice of subalgebras, nilpotent and free Lie algebra.*

In the present paper, the theorems of lattice definability of 2-nilpotent, free and free polynilpotent Lie algebras are proved (for the related problems see [1-6, 11-13]).

In what follows, $\mathcal{K}$ denotes a commutative domain with unit. This property of the ring $\mathcal{K}$ will not be always specified, but sometimes we shall complement it with additional assumptions. L denotes the Lie algebra over $\mathcal{K}$; $\mathcal{L}(L)$ is the lattice of all subalgebras L; $\mathcal{Z}(L)$ is the centre of the Lie algebra L; $\mathcal{N}(A)$ is the normalizer of a subalgebra $A\subseteq L$; $\langle X\rangle$ denotes a subalgebra generated by X. The algebra L is torsion-free if for any $x\in L$, $x\neq 0$, the condition $\alpha x=0$, $\alpha\in\mathcal{K}$, implies $\alpha=0$.

Let $\mathcal{M}$ and $\mathcal{M}_1$ be the linear algebras over the rings $\mathcal{K}$ and $\mathcal{K}_1$, respectively, and $\sigma:\mathcal{K}\to\mathcal{K}_1$ be an isomorphism. A bijection $\mu:\mathcal{M}\to\mathcal{M}_1$ will be called a σ -semilinear quasimorphism if for any $x_1,x_2\in\mathcal{M}$ and $\alpha,\beta\in\mathcal{K}$ there exists $\gamma\in\mathcal{K}_1$ such that

$$\mu(\alpha x_1+\beta x_2)=\sigma(\alpha)\mu(x_1)+\sigma(\beta)\mu(x_2),\quad \mu(x_1x_2)=\gamma\,\mu(x_1)\mu(x_2).$$

If $\gamma=1$, then μ is called a σ -semilinear isomorphism. We say that a lattice isomorphism is induced by a semilinear isomorphism $f:\mathcal{L}(L)\to\mathcal{L}(L_1)$ with respect to σ if $\mu(A)=f(A)$ for any subalgebra $A\subseteq L$.

Proposition 1. *Let the conditions $x^2=0$, $\mathcal{M}^2\neq 0$, $M_1^2\neq 0$ be fulfilled in the linear algebras $\mathcal{M}$ and $\mathcal{M}_1$. Then any isomorphism $f:\mathcal{L}(\mathcal{M})\to\mathcal{L}(\mathcal{M}_1)$ is induced by at most one semilinear isomorphism.*

Proof. The one-generated algebras in $\mathcal{M}$ are considered. We obtain that the semilinear isomorphisms $\varphi,\psi:\mathcal{M}\to\mathcal{M}_1$, which induce *f*, are related by $\varphi(x)=\gamma\psi(x)$, $x\in\mathcal{M}$, where σ is an invertible element from $\mathcal{K}_1$; applying the isomorphisms φ,ψ to the nonzero product of two elements from $\mathcal{M}$, we obtain $\gamma=1$.

Example 1 (see [4, 6, 7]). The lattice isomorphism of a nilpotent Lie algebra may not be induced by a semilinear isomorphism, i.e. an analogue of the fundamental theorem of projective geometry for Lie algebras over the field is not held. Indeed, let ***L*** be a nilpotent Lie algebra

$$L=\langle a,b\rangle,\quad \dim L=3,\ [a,b]=z,$$

f be the onto mapping $\mathcal{L}(L)$ which leaves all two-dimensional subalgebras in their locus, and maps one-dimensional subalgebras arbitrarily but identically modulo the commutant, i.e., for any $x\in L$, $f(\langle x\rangle)=\langle x+z\rangle$. It is clear that f is a lattice automorphism which is not induced by any semilinear isomorphism [4, 6, 8, 9].

A subalgebra $A\subset L$ is called isolated if for any $\ell\in L, \ell\neq 0$, we have $A\cap\langle\ell\rangle\neq 0\Leftrightarrow\langle\ell\rangle\subseteq A$. The isolator $I(A)$ of a subalgebra *A* is called the intersection of all isolated subalgebras containing *A*. The dimension (we will denote it by $\dim L$) of a $\mathcal{K}$ -algebra ***L*** is called the rank of a $\mathcal{K}$ -module ***L***.

Proposition 2 ([6, 8]). *If $f:\mathcal{L}(\mathcal{M})\to\mathcal{L}(\mathcal{M}_1)$ is a lattice isomorphism and $\mathcal{K}$, $\mathcal{K}_1$ are principal ideal domains, then the properties*

(1) f *maps ideals to ideals;*

(2) $f([A,A])=[f(A),f(A)]$ *for any subalgebra* $A\subseteq L$ *;*

(3) $N(f(A))=f(N(A))$ *for any subalgebra* $A\subseteq L$ *;*

(4) f, f^{-1} *maps ideals to ideals*

are related by the diagram

$$(4)\Rightarrow(2)\Leftrightarrow(3)\Rightarrow(1)\Leftarrow(4).$$

Lemma. *Let **L** and L_1 be torsion-free nilpotent Lie algebras over the principal ideal domains $\mathcal{K}$ and $\mathcal{K}_1$; $f:\mathcal{L}(\mathcal{M})\to\mathcal{L}(\mathcal{M}_1)$ be a lattice isomorphism. If $\dim L\geq 3$ and $\mathcal{Z}(L)\geq 2$, then there exists a bijection which is a semilinear isomorphism with respect to an isomorphism $\sigma:\mathcal{K}\to\mathcal{K}_1$ on all abelian subalgebras and $f(A)=\mu(A)$ for any subalgebra $A\subseteq L$.*

The requirement $\dim \mathcal{Z}(L) > 1$ in the conditions of the lemma is essential. To construct the corresponding example, we use an infinite-dimensional minimal non-nilpotent Lie algebra [10].

Example 2. Let ***L*** be the infinite-dimensional solvable Lie algebra over the principal ideals domain *K*,

$$L = [L, L]\lambda\langle x\rangle, \ \ [L, L] = \sum_{i=1}^{\infty}\langle x_i\rangle \ ,^*$$

where the symbol λ denotes semidirect product of Lie algebras;

$$[x, x_1] = 0, \ [x, x_i] = x_{i-1}, \ [x_i, x_j] = 0, \ i, j = 1, 2, \dots .$$

We denote $A_n = \sum_{i=1}^{n}\langle x_i\rangle$. Hence $[L, L] = \bigcup_{n=1}^{\infty} A_n$. Let us construct in ***L*** the increasing series of ideals

$$0 \subset \langle x_1, x\rangle = B_1 \subset A_2\lambda\langle x\rangle = B_2 \subset \cdots \subset A_i\lambda\langle x\rangle B_i \subset \cdots \subset L = \bigcup_{k=1}^{\infty} B_k \, .$$

We are to show that for each n, the lattice $\mathcal{L}(B_n)$ admits an automorphism σ_n which is identical on $\mathcal{L}(A_n)$ and not inducible by a semilinear isomorphism on abelian subalgebras. Indeed, for $n = 1$ we have an abelian subalgebra $\langle x_1\rangle \cup \langle x\rangle = B_1$, on which we can construct a lattice automorphism that leaves B_1 in locus and is not induced by a semilinear isomorphism: any one-to-one correspondence between one-dimensional subalgebras defines the automorphism of $\mathcal{L}(B_1)$. We define by the induction the automorphism $\sigma_n : \mathcal{L}(A_n\lambda\langle x\rangle) \to \mathcal{L}(A_n\lambda\langle x\rangle)$ which is identical on all n-dimensional subalgebras except $A_{n-1}\lambda\langle x\rangle$. There exists by induction an automorphism σ_{n-1} of the lattice $\mathcal{L}(A_{n-1}\lambda\langle x\rangle)$, which is identical on A_{n-1} and is not inducinle by a semilinear isomorphism on abelian subalgebras. It is easy to see that the construcred automorphisms σ_n, $n = 1, 2, \dots$, can be chosen so that the restrictions σ_n on A_n coincide with σ_{n-1}. If now the automorphism σ is defined as the union of all σ_n, i.e. $\sigma = \bigcup_{n=1}^{\infty}\sigma_n$, then we obtain the sought automorphism $\mathcal{L}(L)$.

A subalgebra $X \subseteq L$ is called quasi-ideal if for any subalgebra $Q \subseteq L$ we have $X \cup Q = X + Q$. For modular subalgebras and quasi-ideals see [7, 9, 10, 12, 13].

The torsion-free Lie algebra ***L*** over a ring *K* is called almost abelian if $L = A\lambda\tilde{R}$, where *A* is an abelian ideal and $\tilde{R}$ is locally cyclic, i.e. each finitely generated subalgebra is one-generated and for any $a \in A$, $x \in \tilde{R}$ there exists $n(x) \in \mathcal{K}$ such that

$$n(x)[x, a] = ma, \ m \in \mathcal{K} .$$

In almost abelian Lie algebra, each submodule is a subalgebra. The latter fact is equivalent to the definition of almost abelian Lie algebras [7] (in this connection, see also [8, 11-13]).

Theorem 1. *Let $f : \mathcal{L}(L) \to \mathcal{L}(L_1)$ be a lattice isomorphism, where L and L_1 are torsion-free nilpotent Lie algebras of class 2 over the principal ideal domains $\mathcal{K}$ and $\mathcal{K}_1$, respectively. If $\dim L \neq 3$, then L and L_1 are semilinear isomorphic.*

The proof of Theorem 1 is based on the above Lemma 1 and Proposition 2.

Remark 1. If $\mathcal{K} = Z_p$, then the condition $\dim L > 3$ is unnecessary because for $\dim L = 2, 3$ the lattice $\mathcal{L}(L)$ has orders p^2 and p^3.

Remark 2. If the algebras L and L_1 are defined over one and the same ring, then the restriction $\dim L > 3$ is also unnecessary because for $\dim L = 3$, L and L_1 are free 2-nilpotent Lie algebras and $L \cong L_1$; the case $\dim L = 2$ is trivial.

Proposition 3. *Let $f : \mathcal{L}(L) \to \mathcal{L}(G)$ be a lattice isomorphism and the Lie algebra L have a series of isolated modular subalgebras*

$$0 = L_0 \subset L_1 \subset \cdots \subset L_\beta = L \tag{1}$$

such that the intervals $|L_\alpha, L_{\alpha+1}|$ are modular lattices. If $L, G \in \Sigma$, where Σ is a closed class with respect to homomorphisms, such that for $\forall X \in \Sigma$ the lattice $\mathcal{L}(X)$ is modular if and only if X is almost abelian, then f maps isolated quasi-ideals to isolated quasi-ideals.

Theorem 2. *Let $f : \mathcal{L}(L) \to \mathcal{L}(G)$ be a lattice isomorphism, L and G be the Lie algebras over the principal ideals domains $\mathcal{K}$ and $\mathcal{K}_1$, respectively. If L is a free (nonabelian) polynilpotent Lie algebra and $\dim L \neq 3$, then $\mathcal{K} \cong \mathcal{K}_1$ and $L \cong G$.*

Theorem 3. *Let L and G be the Lie algebras over the principal ideal domains $\mathcal{K}$ and $\mathcal{K}_1$, respectively; $f : \mathcal{L}(L) \to \mathcal{L}(G)$ be a lattice isomorphism. If L is a free Lie algebra, then $\mathcal{K} \cong \mathcal{K}_1$ and $L \cong G$.*

The proofs of the Theorems 2 and 3 are based on Propositions 2 and 4.

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ნილპოტენტური და თავისუფალი ლის ალგებრების მესერული იზომორფიზმები

ა. ლაშხი

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(წარმოდგენილია აკადემიკოს ხ. ინასარიძის მიერ)

ნაშრომში შესწავლილია ნილპოტენტური და თავისუფალი ლის ალგებრების მესერული იზომორფიზმები. $\mathcal{K}$ აღნიშნავს კომუტატურ მთელობის არეს ერთეულით. L ლის ალგებრა $\mathcal{K}$. $\mathcal{L}(L)$ აღნიშნავს L-ის ქვეალგებრათა მესერს. თუ $\mathcal{M}$ და $\mathcal{M}_1$ წრფივი ალგებრებია, შესაბამისად

$\mathcal{K}$ **და** $\mathcal{K}_1$ **რგოლებზე, ხოლო** $\sigma : \mathcal{K} \to \mathcal{K}_1$ **რგოლური იზომორფიზმია, მაშინ** $\mu : \mathcal{M} \to \mathcal{M}_1$ **ბიექციას ეწოდება** σ **-ნახევრადწრფივი კვაზიიზომორფიზმი, თუ**

$$\mu(\alpha x_1 + \beta x_2) = \sigma(\alpha)\mu(x_1) + \sigma(\beta)\mu(x_2), \quad \mu(x_1 x_2) = \lambda\mu(x_1)\mu(x_2), \ \forall x_1, x_2 \in \mathcal{M}, \ \ \forall \alpha, \beta \in \mathcal{K}.$$

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