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Mathematics

On the Estimation of a Maximum Likelihood of Truncated Exponential Distributions

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ABSTRACT. The problem of estimation of parameters of truncated exponential distribution by the maximum likelihood method is studied. It is shown that maximum likelihood equation for truncated exponential distribution has a unique solution which gives an asymptotic effective estimator of the parameter. *© 2013 Bull. Georg. Natl. Acad. Sci.*

Key words: *truncation, exponential distribution, maximum likelihood estimate.*

Let X be a truncated exponentially distributed random value with density

$$f(x;\theta,\alpha,\beta)=\begin{cases}\dfrac{\theta e^{-\theta x}}{e^{-\alpha\theta}-e^{-\beta\theta}}, & \alpha<x\le\beta\\ 0, & x\le\alpha,\ x>\beta\end{cases}, \tag{1}$$

where $\alpha<\beta$ and α, β, and θ are the unknown parameters. Let $X_1,X_2,\ldots,X_n$ be a random sampling of size n taken from the truncated exponential distributions given by (1). It is required to estimate α, β, and θ by these observations. We do this by applying the maximum likelihood estimator.

A likelihood function has the form

$$L(x;\theta,\alpha,\beta)=\theta^n\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)^{-n}\cdot\exp\left(-\theta\sum_{i=1}^{n}x_i\right)=\theta^n\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)^{-n}\cdot\exp\left(-n\theta\overline{X}\right). \tag{2}$$

The logarithm of likelihood functions is defined and differentiated on the open interval $(0,\infty)$, whereas a maximum value, if it exists, takes place at a stationary point at which

$$\frac{\partial\ln L(x;\theta,\alpha,\beta)}{\partial\theta}=0,$$

but does not take place at a boundary point lying on the interval $(0,\infty)$.

Lemma [1]. *Let the following conditions be fulfilled:*

(1) *the distribution function* $F(x,\theta)$ *is continuous with respect to both variables and has the continuous derivative* $f(x,\theta)=\dfrac{\partial F(x,\theta)}{\partial x}$;

(2) *the likelihood function* L *has the absolute maximum* $\theta=\overline{\theta}_n$.

Then $\overline{\theta}_n$ *is an asymptotically consistent and asymptotically effective estimate of the true value of the parameter* θ.

The logarithm of likelihood functions is written as

$$\ln L(x;\theta,\alpha,\beta)=n\ln\theta-n\ln\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)-n\theta\overline{X}.$$

Consider the equation

$$\frac{\partial\ln L(x;\theta,\alpha,\beta)}{\partial\theta}=0,$$

$$\frac{\partial\ln L(x;\theta,\alpha,\beta)}{\partial\theta}=\frac{n}{\theta}-\frac{n\left(\beta\cdot e^{-\theta\beta}-\alpha\cdot e^{-\theta\alpha}\right)}{e^{-\alpha\theta}-e^{-\beta\theta}}-n\overline{X}=0, \tag{3}$$

$$\frac{1}{\theta}-\left(\beta\cdot e^{-\beta\theta}-\alpha\cdot e^{-\alpha\theta}\right)\cdot\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)^{-1}-\overline{X}=0.$$

Let us investigate this equation for $\theta\to 0$. Note that for $\theta\to 0$ we have $e^{-\alpha\theta}-e^{-\beta\theta}\to 0$.

According to l'Hospital's rule, we obtain

$$\lim_{\theta\to 0}\left(\frac{1}{\theta}-\left(\beta e^{-\beta\theta}-\alpha e^{-\alpha\theta}\right)\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)^{-1}\right)=\lim_{\theta\to 0}\frac{\frac{e^{-\alpha\theta}-e^{-\beta\theta}}{\beta e^{-\beta\theta}-\alpha e^{-\alpha\theta}}-\theta}{\frac{\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)\cdot\theta}{\beta e^{-\beta\theta}-\alpha e^{-\alpha\theta}}}=$$

$$=\lim_{\theta\to 0}\frac{-\theta\left(-\beta^2e^{-\beta\theta}+\alpha^2e^{-\alpha\theta}\right)}{\theta\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)+\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)}=$$

$$=\lim_{\theta\to 0}\frac{-\theta\left(\beta^3e^{-\beta\theta}-\alpha^3e^{-\alpha\theta}\right)-\left(-\beta^2e^{-\beta\theta}+\alpha^2e^{-\alpha\theta}\right)}{\left(\beta e^{-\beta\theta}-\alpha e^{-\alpha\theta}\right)+\theta\left(\beta e^{-\beta\theta}-\alpha e^{-\alpha\theta}\right)-\alpha e^{-\alpha\theta}+\beta e^{-\beta\theta}}=$$

$$=\frac{\beta^2-\alpha^2}{2(\beta-\alpha)}=\frac{\beta+\alpha}{2}, \tag{4}$$

i.e.

$$\lim_{\theta\to 0}\frac{\partial\ln L(x;\theta,\alpha,\beta)}{\partial\theta}=\frac{\beta+\alpha}{2}-\overline{X}.$$

Analogously, for $\theta\to\infty$ we have $\lim\limits_{\theta\to\infty}\dfrac{\partial\ln L(x;\theta,\alpha,\beta)}{\partial\theta}=-\overline{X}$. If $0<\overline{X}<\dfrac{\beta+\alpha}{2}$, then there exists a stationary point, say θ^*, such that

$$\left.\frac{\partial \ln L(x;\theta,\alpha,\beta)}{\partial\theta}\right|_{\theta=\theta^*}=0\,.$$

Let us check that at this point the second derivative is negative. We write the second derivative in the form

$$\frac{\partial^2 \ln L(x;\theta,\alpha,\beta)}{\partial\theta^2}=$$

$$=-\frac{n}{\theta^2}-\frac{n\left(\alpha^2e^{-\alpha\theta}-\beta^2e^{-\beta\theta}\right)\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)}{\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)^2}-\frac{\left(\beta e^{-\beta\theta}-\alpha e^{-\alpha\theta}\right)\left(\beta e^{-\theta\beta}-\alpha e^{-\theta\alpha}\right)}{\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)^2}=$$

$$=\frac{n\cdot e^{-\theta(\beta+\alpha)}\cdot(\beta-\alpha)^2}{\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)^2}-\frac{n}{\theta^2}=n\left(\frac{e^{-\theta(\beta+\alpha)}\cdot(\beta-\alpha)^2}{\left(e^{-\alpha\theta}\right)^2\left(1-e^{-\theta(\beta-\alpha)}\right)^2}-\frac{1}{\theta^2}\right).$$

If we use the elementary inequality $1-e^x<-x$, then for $x\neq 0$ and $\frac{e^{-\theta\beta}}{e^{-\theta\alpha}}<1$ we obtain

$$\frac{\partial^2 \ln L(x;\theta,\alpha,\beta)}{\partial\theta^2}<n\left(\frac{e^{-\theta(\beta+\alpha)}\cdot(\beta-\alpha)^2}{\left(e^{-\alpha\theta}\right)^2\cdot\theta^2(\beta-\alpha)^2}-\frac{1}{\theta^2}\right)=n\left(\frac{e^{-\theta\beta}}{e^{-\theta\alpha}\cdot\theta^2}-\frac{1}{\theta^2}\right)<0\,.$$

Therefore equation (3) has a unique solution θ^*, $0<\theta^*<\infty$ and it is unique at a maximum point. A maximum likelihood estimate $\hat{\theta}$ for the parameter θ is written in the following form

$$\hat{\theta}=\begin{cases}\theta^* \text{ if } 0<\bar{X}<\dfrac{\beta+\alpha}{2}\\ \text{not exist if } \bar{X}\geq\dfrac{\beta+\alpha}{2}\end{cases}.$$

In view of the assertion of Lemma, the following theorem is valid.

Theorem. *Assume that we have the sample* $X_1,X_2,\ldots,X_n$ *of random values which are distributed according to law* (1) *where* α, β *and* θ *are the unknown parameters.*

If $0<\bar{X}<\frac{\beta+\alpha}{2}$, *then the maximum likelihood estimate for* θ *exists and is the unique root of the equation*

$$\frac{1}{\theta}-\left(\beta e^{-\beta\theta}-\alpha e^{-\alpha\theta}\right)\left(e^{-\alpha\theta}-e^{-\beta\theta}\right)^{-1}-\bar{X}=0\cdot \tag{5}$$

This estimate is consistent and asymptotically effective.

Also, $\alpha=X_{(1)}=\min(X_1,\ldots,X_n)$, $\beta=X_{(n)}=\max(X_1,\ldots,X_n)$.

Remark. If $\alpha = 0$ and $\beta = \infty$, then from (5) we obtain the classical case

$$\theta = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i}.$$

Computerized simulation of an exponential distribution with parameters $\alpha = 1$, $\beta = 2$ and $\theta = 2$ was carried out. For the sample of size $n = 10000$, we obtained the estimate $\overline{X} = 1{,}344$, $\hat{\theta} = 2{,}005$.

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