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Mathematics

Equilibria of Constrained Point Charges

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ABSTRACT. We discuss analogs of the famous Maxwell conjecture on the number of equilibria of point charges in certain situations where the positions of charges are subject to quadratic constraints. Two types of quadratic constraints are considered in some detail: point charges placed at the vertices of polygonal linkage, and point charges confined to a circle. We show that the Coulomb potential in both cases is a Morse function on the corresponding moduli space and present several results on the number and Morse indices of its critical points. Detailed results are obtained if the number of charges does not exceed four. For quadrilateral linkage, we establish that the number of equilibria does not exceed eight. As a by-product, we show that any convex configuration of such a linkage is an equilibrium of Coulomb potential for some collection of charges at the vertices. For three charges on the circle, we give a geometric characterization of configurations which are equilibria of Coulomb potential for some collection of charges. *© 2013 Bull. Georg. Natl. Acad. Sci*

Key words: *point charge, Coulomb potential, equilibrium, polygonal linkage, configuration space, moduli space, Morse function, critical point.*

1. Equilibrium configurations of point charges with Coulomb interaction have been extensively discussed as models of real physical systems [1, 2], and in connection with the famous Maxwell conjecture [3, 4]. As was recently realized by the author, this paradigm leads to interesting developments in the setting where the charges are placed at the vertices of polygonal linkage [5]. Similar situations where the positions of charges are subject to certain geometric or analytic constraints were discussed in [1, 2].

Motivated by these developments, we study the equilibria of point charges subject to certain quadratic constraints, which generalizes the settings of [2] and [5]. A characteristic feature of our approach is that we consider Coulomb potential as a function on the moduli space naturally associated with the constraints considered. An important circumstance is that, for quadratic constraints, the topology of the moduli space can be effectively investigated using the methods of [6], which yields considerable information on the critical points of Coulomb potential.

Another peculiarity of the point of view accepted in this note is that we apply our results to the problem of controlling the shape of charged configuration by the values of charges. This is conceptually close to the setting of necklaces with interacting beads discussed in [2] but the problem of shape control has not been discussed in [2]. An impetus for considering this problem was given by some recent research on the dynamics of nano-systems presented in [7].

Recall that the famous conjecture of J.C.Maxwell states that the number of Coulomb equilibria of *n* point charges in generic position in $\mathbb{R}^3$ does not exceed $(n-1)^2$ [1]. This conjecture remains unproven even for $n=3$ and the best established estimate for n=3 is 12 [2]. Being classical and intriguing, this conjecture gained a lot of attention of researchers [2]. In particular, various modifications and special cases of Maxwell conjecture have been considered, which led to a number of interesting mathematical results obtained by topological [3] and analytical methods (cf. [4]).

Along these lines, we consider *systems of points with quadratic constraints*, give a few natural examples of such systems and discuss analogs of Maxwell conjecture in this context. We illustrate this setting by considering *vertex-charged polygonal linkages* and *point charges on the circle*, present several related results for linkages with small number of vertices and discuss a few promising topics naturally arising in the framework of our setting. Our approach relies on the topological results on intersections of quadrics [6] and signature formulae for the topological invariants of functions on moduli spaces [8].

2. We begin with describing the definitions and the main paradigm. Recall that the *Coulomb potential* of a system V of unit charges q_i placed at the points $v_i \in \mathbb{R}^3$ is a rational function on $\mathbb{R}^3$ defined by the formula $\Psi_V(P) = \Sigma\, q_i\, (d(P,v_i))^{-1}$, where $P \in \mathbb{R}^3$ and $d(P,v_i)$ denotes the Euclidean distance between P and v_i. The *electrostatic energy* of V is defined as $E_s(V) = \Sigma\, q_i q_j\, (d(v_i,v_j))^{-1}$, where the sum is taken over all pairs of nonequal indices.

Consider now the collection S(N, G) of all N-tuples of points $v_i \in \mathbb{R}^3$ such that their coordinates satisfy a set of quadratic equations G. In such a situation we will speak of *system of points with quadratic constraints* (SPQC). Factoring S(N, G) by the diagonal action of the group of orientation preserving isometries of $\mathbb{R}^3$ we obtain the moduli space M(N,G) of the given SPQC.

We now wish to consider the system S(Q, G) of point charges q_i placed at the points of a SPQC and call it a *system of charges with quadratic constraints* (SCQC). In this situation, the electrostatic energy of SCQC naturally defines a rational function $E_Q : M(N,G) \to \mathbb{R}$ on the moduli space M(N, G), which is the main object of our interest in this paper. In particular, the minima of electrostatic energy correspond to equilibria of a system of point charges satisfying the given quadratic constraints.

It appears also useful to consider all critical points of this function. Then an analog of the aforementioned Maxwell problem can be formulated as the search for an exact upper bound for the number of critical points of E_Q in M(N, G) valid for all systems of charges Q. An important circumstance is that one can effectively compute the homology groups of the moduli space M(N,G) using the methods of [6]. It turns out that generically E_Q is a Morse function on M(N,G) and so one can obtain certain information on its critical points from the homology groups of M(N,G). This is the main paradigm for the considerations presented in the sequel.

This paradigm is obviously applicable to a number of well-known settings such as the moduli space of polygonal linkage [5], point charges on a circle [1, 2], configuration space of N points on an ellipse, conformation spaces of certain organic molecules, families of poristic bicentric polygons [9] and many more.

To illustrate our approach we discuss in some detail the situation where point charges are placed at the vertices of polygonal linkage which was studied in [5].

3. Recall that a polygonal linkage L is defined by a *k*-tuple of positive numbers l_i called *sidelengths* of L. In the case of a closed polygonal linkage it is always assumed that each of the sidelengths is not greater than the sum of all other ones. A polygonal linkage is called *regular* if all sidelengths are equal. The *planar* *configuration space* C(L) of a polygonal *k*-chain L is defined as the collection of all k-tuples of points v_i in Euclidean plane such that the distance between v_i and v_{i+1} is equal to l_i, where it is assumed that $v_{k+1} = v_1$. Each such collection of points is called a *configuration* of L. Since the distance condition is quadratic, a polygonal linkage gives an example of SPQC.

A configuration is called *convex* if the corresponding polygon is convex. Factoring C(L) over the natural diagonal action of SO(2) one obtains the *(planar) moduli space* M(L). A subset of M(L) formed by the convex configurations will be denoted by $M^c(L)$. Moduli spaces, as well as configuration spaces, are endowed with the natural topologies induced by Euclidean metric.

It is easy to see that the planar moduli space can be identified with the subset of configurations such that $v_1 = (0,0)$, $v_2 = (l_1,0)$. It is well known that, for a closed *k*-chain, the moduli space has a natural structure of compact orientable real-algebraic set of dimension $k - 3$. Let us say that a polygonal linkage is *degenerate* if it has an *aligned configuration*, i.e., a configuration where all vertices lie on the same straight line. It is well known that this happens if and only if there exists a k-tuple of "signs" $s_i = \pm 1$ such that $\Sigma s_i l_i = 0$. The moduli spaces M(L) of polygonal linkage L are smooth (do not have singular points) if and only if L is nondegenerate [8].

According to our paradigm, we fix a polygonal linkage L and system of charges Q and consider E_Q as a function on the moduli space M(L). In this paper, we only deal with the case of *quadrilateral linkage*.

So let L = L (a, b, c, d) be a nondegenerate quadrilateral linkage with pairwise non-equal lengths of the sides. For brevity, such a linkage will be called *admissible*. For each configuration V of L, and a system of charges Q, let us consider its Coulomb energy $E_Q(V)$. Since L does not have configurations with coinciding vertices, E_Q is a smooth (infinitely differentiable) function on M(L). So one may consider its critical points which in fact correspond to the *equilibria* of Q-charged linkage L subject only to electrostatic forces between its vertices.

In this setting, one may wish to investigate three natural problems arising as generalizations of the two problems formulated in [5]: (P1) for a given admissible linkage L and system of charges Q, find the number of equilibria of E_Q in M(L); (P2) for a given admissible linkage L, find the maximal possible number of equilibria of E_Q over the set all charges Q; (P3) find the maximal possible number of equilibria of E_Q over the set of all admissible quadrilateral linkages L and all charges Q.

These problems obviously have many modifications. For example, in certain situations it is natural to consider only system of charges of the same sign or just a system of equal charges.

Notice a conceptual analogy of (P3) in the case of equal charges with the Maxwell conjecture [3]. However, an essential difference is that here we consider the equilibria of the linkage itself and not the equilibria of its Coulomb potential in the ambient space. In such a setting, the problem acquires several new aspects, which lead to the following results in the spirit of [8]. First of all, problem (P1) can be solved using our general approach based on signature formulae for topological invariants [8] which generalizes a similar result given in [5].

Theorem 1. *For an admissible quadrilateral linkage L and system of charges Q, the number of equilibria of E_Q can be calculated as the signature of a quadratic form with the coefficients algebraically expressible through the sidelengths of L and values of charges.*

In fact, the same result holds for many classes of SCQC satisfying some mild conditions of genericity (transversality), in particular, for linkages with an arbitrary number of sides. All these results follow by applying the signature formula for the Euler characteristic to E_Q and the polynomial system for the equilibria of SCQC obtained by the method of Lagrange multipliers. Indeed, equilibria correspond to the real solutions to the arising Lagrange system which can be counted by the signature formula from [8]. However, a rigorous formulation of the general result requires more concepts and space so it is postponed for future publications.

Using the methods of [8] one can also show that, generically, all equilibria are in fact nondegenerate in the sense of Morse theory.

Theorem 2. *For a generic admissible quadrilateral linkage L and system of charges, E_Q is a Morse function on M(L).*

The proof is obtained by analyzing the constrained optimization problem for the Coulomb potential. To this end we take the lengths (x,y) of diagonals of configuration as coordinates on M(L) and consider the extended Hessian matrix of Lagrange function in these coordinates. Due to the simple form of E_Q in these coordinates, the determinant of extended Hessian can be computed explicitly, which yields the result.

There is good evidence that an analogous result holds for linkages with an arbitrary number of sides and, more generally, for wide classes of SPQC. However, the method outlined above is not realistic in the general case, so one should think of a proof based on transversality theorems.

Results of such kind are useful because they enable one to estimate the number of equilibria using Morse inequalities and similar topological tools [6]. For quadrilateral linkages, this is not too interesting since M(L) has very simple topology, but this observation may be helpful in the general setting of SCQC since the topology of their moduli spaces can be described using the methods of [6].

After having found the polynomial system real solutions to which give the equilibria of potential in the moduli space, one can compute the bifurcation diagram of this system in the space of all parameters (sidelengths and charges) and find the number of equilibria in each component of its complement by the aforementioned signature formula. Realizing this program for quadrilateral linkages we arrive at the following result generalizing a similar result from [5].

Theorem 3. *For any admissible linkage L and system of charges Q, E_Q has no more than eight critical points on M(L).*

This result may be considered as the first step towards proving the analog of Maxwell conjecture in the setting of polygonal linkages [3].

4. The above results on electrostatic equilibria of linkages have the following curious application in the spirit of control theory which can hopefully serve as a paradigm for similar developments in the context of general systems of charges with quadratic constraints. Consider an admissible quadrilateral linkage L as above and place a system Q of positive charges at its vertices. Among the critical points of E_Q the global minima are especially important since they give the *stable equilibria* of the linkage subject to only electrostatic forces.

As is geometrically obvious and can be easily verified, given a non-convex planar configuration one can increase both of its diagonals simultaneously by deforming the linkage. Thus the global minima of Coulomb potential always belong to $M^c(L)$. In fact, one has the following two results which have been proven jointly with G.Panina and D.Siersma.

Theorem 4. *For an admissible quadrilateral linkage L with arbitrary positive charges at its vertices, the global minimum of E_Q on M(L) is unique.*

This result suggests the idea of controlling the shape of L by changing the charge at one of the vertices of L. More precisely, we fix the positions of the first two vertices of L, place unit positive charges at all vertices except the first one placed at the origin, and permit ourselves to change the value $q > 0$ of charge at the first vertex. By Theorem 4, for each $q > 0$ we have a single stable equilibrium of E_Q, i.e. a well-defined point in $M^c(Q)$ which we denote by V(q). A natural question now is whether this mapping is surjective as a mapping from $\mathbf{R}_+$ into $M^c(L)$. A positive answer to this question would mean that we may force the linkage to take any convex shape from $M^c(L)$ by choosing a proper value of q.

Theorem 5. *For an admissible quadrilateral linkage L, the mapping from* $\mathbf{R}_+$ *into* $M^c(L)$ *defined by sending q to* $V(q)$ *is surjective on the interior of* $M^c(L)$.

Outline of the proof. First of all, using Lagrange method it is easy to see that the configuration with the lengths of diagonals equal to (x_o, y_o) is the global minimum of E_s when the charge at the first vertex is equal to $q=(x_o)^2(y_o)^{-2}T_x(T_y)^{-1}$, where the subscripts denote partial derivatives and both y_o and T_y are non-zero. If $T_y = 0$ one uses an analogous relation with the roles of x and y exchanged (notice that the lengths of both diagonals are nonzero for an admissible linkage). It only remains to show that the obtained value of q is indeed positive. This can be proven by analyzing the implicit functions of the form *y(x)* and *x(y)* obtained from the Euler four point relation (see, e.g., [5]). A simple argument implies that, for a convex configuration, both partial derivatives have the same sign, which completes the proof.

This result means that convex configurations of charged quadrilateral Q as above can be completely controlled by the value of charge at just one of its vertices. Similar problems obviously make sense for linkages with an arbitrary number of sides and in the context of SPQC described above. However, their investigation appeared rather difficult already for pentagons.

5. In conclusion we briefly discuss similar topics in the situation where the charges are confined to stay on the unit circle, which is obviously another example of SCQC. For physical reasons, in this situation it is reasonable to consider charges of the same sign.

Consider first the case of three positive charges. It is easy to see that each configuration V which is an equilibrium for a certain system of three positive charges, has the following geometric property: the diameter passing through each charge separates the two remaining charges. For brevity, let us call it *separation property.* In fact, it is easy to show that, for three charges, this property is a criterion.

Theorem 6. *A configuration V of three points on the unit circle is an equilibrium of three positive charges placed at these points if and only if V has the separation property.*

We were unable to prove that the separation condition is sufficient if there are more than three charges. However, it seems very likely that an analogous criterion is valid for any number of positive charges on the circle. It would be also interesting to investigate the case of three charges on an ellipse which fits into the same paradigm.

In conclusion we add that analogs of the above results and conjectures make sense for general systems of points with quadratic constraints. The author intends to continue the study of these problems in the spirit of the paradigm described above.

მათემატიკა

შეზღუდვების მქონე წერტილოვანი მუხტების წონასწორობები

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(წარდგენილია აკადემიკოს რ. გამყრელიძის მიერ)

ნაშრომში განხილულია მაქსველის ცნობილი ჰიპოთეზის ანალოგები კვადრატული შეზღუდვების მქონე წერტილოვანი მუხტების შემთხვევაში. უფრო დეტალურად განხილულია ორი ტიპის შეზღუდვები: წერტილოვანი მუხტები სახსრული მრავალკუთხედის წვეროებში და წერტილოვანი მუხტები წრეწირზე. დადგენილია, რომ ამ შემთხვევებში კულონის პოტენციალი განსაზღვრავს მორსის ფუნქციას კონფიგურაციულ სივრცეზე და მოყვანილია რამდენიმე შედეგი ამ ფუნქციის კრიტიკულ წერტილთა რაოდენობის და მორსის ინდექსების შესახებ. საკმაოდ დეტალური შედეგები მიღებულია იმ შემთხვევაში, როდესაც მუხტების რაოდენობა არ აღემატება ოთხს. კერძოდ, ნაჩვენებია, რომ სახსრული ოთხკუთხედისთვის წონასწორობების რაოდენობა არ აღემატება რვას და ასეთი ოთხკუთხედის ნებისმიერი ამოზნექილი კონფიგურაცია მიიღება როგორც მისი დამუხტული წვეროების წონასწორობა. ანალოგიური შედეგი მიღებულია სამი მუხტისთვის წრეწირზე.

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