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Mathematics

On the Convergence of the Crank-Nicolson Semidiscrete Scheme for an Evolutionary Equation in the Banach Space

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ABSTRACT. The Crank-Nicolson semidiscrete scheme is considered for an evolutionary equation with a linear unbounded closed operator A in the Banach space. It is proved that if (a) the spectrum of the operator A is contained in a symmetrical open sector with an angle opening less than π, lying in the right-hand half-plane; (b) for any point z ($z \neq 0$) not belonging to this sector, the resolvent norm is not greater than $c/|z|$; (c) the second derivative of a solution satisfies the Lifshitz condition. Then the error of an approximate solution is not greater than $c(\varepsilon_0 \ln \tau^{-1} + \varepsilon_1 + \tau^2)$, where τ is a grid step, while ε_0 and ε_1 are the disturbance of the initial vector and the right-hand part, respectively. *© 2013 Bull. Georg. Natl. Acad. Sci.*

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As is known, various initial boundary value problems for nonstationary equations with partial derivatives can be reduced to the Cauchy problem for an evolutionary equation in the Banach space. One of the methods of solution of these problems is the semidiscretization method. This method has an advantage that the system obtained as a result of its application can be solved, for instance, by the finite difference method and the subsequent discretization of derivatives is carried out with respect to spatial variables. Other methods, including analytical ones, can also be used.

Questions connected with the construction and investigation of approximate solution algorithms of evolutionary problems are considered for example in the well-known books by S. K. Godunov and V. S. Ryaben'kii [1], G. I. Marchuk [2], R. Richtmayer and K. Morton [3], A. A. Samarski [4], N. N. Ianenko [5]. We also refer to the works by H. A. Alibekov and P. E. Sobolevski [6], A. E. Polichek and P. E. Sobolevski [7], M.Crouzeix [8], M. Crouzeix and P.-A. Raviart[9] and M.-N. Le Rouxe [10] dedicated to the approximate solution of the Cauchy problem for an abstract parabolic equation. The results obtained in these papers embrace a sufficiently wide class of evolutionary problems.

In the present paper, our investigation relies essentially on the methods developed in the above-mentioned works and in the monograph [11] by the first author.

We consider here the semidiscrete Crank-Nicolson scheme for an evolutionary equation with a linear unbounded closed operator A in the Banach space. It is proved that if (a) the spectrum of the operator A is contained in a symmetrical open sector with an angle opening less than π, lying in the right-hand half-plane, (b) for any point z ($z \neq 0$) not belonging to this sector, the resolvent norm is not greater than $c/|z|$, (c) the second derivative of a solution satisfies the Lifshitz condition, then the error of an approximate solution is not greater than $c(\varepsilon_0 \ln \tau^{-1} + \varepsilon_1 + \tau^2)$, where τ is a grid step, while ε_0 and ε_1 are the disturbance of the initial vector and the right-hand part, respectively.

It should be noted that in Serdyukova's paper [12], the stability of linear difference schemes with constant coefficients, is investigated in a concrete Banach space, in particular in the space C. That paper was one of the pioneer works on the study of difference schemes in the Banach space. We also want to refer to Sobolevski's paper [13], where a logarithmic estimate is given (without proving it) for the resolvent operators of the Crank-Nicolson scheme. It is pointed out that this estimate is proved by the Cauchy-Riesz formula if the initial operator is strongly positive. We further refer to [14], where an explicit estimate of the error of the Crank-Nicolson scheme is obtained in the Hilbert space under the assumption that the initial operator is self-adjoint and positively defined. In the same paper, under the same assumptions for the operator, a lemma is proved, by means of which the obtained results are easily extended to the Banach space.

1. Statement of the Problem and the Formulation of the Basic Theorem

In the Banach space X, we consider the evolutionary problem

$$\frac{du(t)}{dt} + Au(t) = f(t), \quad t \in]0,T], \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where $(-A)$ is the generating operator of a strongly continuous semigroup $\exp(-tA)$, $t \geq 0$; $f(t)$ is a continuously differentiable abstract function with values from X; u_0 is a given vector from X, $u(t)$ is the sought function.

Let us introduce $[0,T]$ the grid $t_k = k\tau$, $k = 0,1,\ldots,n$, with step $\tau = T/n$. For problem (1.1)-(1.2) we consider the Crank-Nicolson scheme

$$\frac{u_k - u_{k-1}}{\tau} + A\frac{u_k + u_{k-1}}{2} = f_{k-\frac{1}{2}}, \quad k = 1,\ldots,n, \tag{1.3}$$

where $f_{k-\frac{1}{2}} \approx f\left(t_{k-\frac{1}{2}}\right)$.

The following theorem is true.

Theorem 1.1. *Assume that the following conditions are fulfilled:*

(a) The solution $u(t)$ of problem (1.1)-(1.2) is twice continuously differentiable and $u''(t)$ satisfies the Lifshitz condition;

(b) A is a linear, densely defined closed operator in the Banach space X, whose spectrum is wholly contained in the sector

$$S = \left\{ z : |\,arg(z)\,| < \varphi_0,\ 0 < \varphi_0 < \frac{\pi}{2} \right\}$$

and the condition

$$\left\| (zI - A)^{-1} \right\| \le \frac{c_0}{|\,z\,|},\quad c_0 = const > 0$$

is fulfilled for any z, $(z \ne 0)$ *not belonging to S. Then the estimate*

$$\|u(t_k) - u_k\| \le c\left(\ln\frac{et_k}{\tau}\|z_0\| + t_k \max_{1\le i\le k}\left\| f\left(t_{i-\frac{1}{2}}\right) - f_{i-\frac{1}{2}} \right\| + \tau^2 t_k \right),\quad k = 1,\ldots,n,$$

is valid, where $z_0 = u_0 - u(t_0)$, $c = const > 0$.

To prove this theorem, we need some auxiliary statements, which, in our opinion, are of independent interest.

2. Auxiliary Statements

Lemma 2.1. *Let us assume that the operator A satisfies the conditions of Theorem 1.1. Then the following estimate is valid*

$$\left\| \frac{\tau}{2}A\left(I - \frac{\tau}{2}A\right)^k \left(I + \frac{\tau}{2}A\right)^{-(k+1)} \left(I + \tau A\right)^{-j} \right\| \le \frac{c_1(\lambda)}{k+j}, \tag{2.1}$$

where k and j are natural numbers, $\tau > 0$,

$$\begin{aligned} c_1(\lambda) &= \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} + \frac{1}{\lambda\lambda_1} \right)\frac{c_0}{2\pi}, \\ \lambda_1 &= \min\left(16\lambda^2, \left(\ln\frac{9}{4\lambda^3} \right)^{-1} \right), \\ \lambda &= \cos(\varphi),\ \varphi_0 \le \varphi < \frac{\pi}{2}. \end{aligned}$$

Proof. Applying the Danford-Taylor integral (see e.g. [15]), we have

$$\begin{aligned} &\frac{\tau}{2}A\left(I - \frac{\tau}{2}A\right)^k \left(I + \frac{\tau}{2}A\right)^{-(k+1)} \left(I + \tau A\right)^{-j} \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{z(1-z)^k}{(1+z)^{k+1}(1+2z)^j}\left(zI - \frac{\tau}{2}A \right)^{-1} dz, \end{aligned} \tag{2.2}$$

where Γ is the boundary of a sector $|\,arg(z)\,| < \varphi$ (the integral is taken along the positive direction).

By virtue of Theorem 1.1 we have

$$\left\| \left(zI - \frac{\tau}{2} A \right)^{-1} \right\| = \frac{2}{\tau} \left\| \left(\frac{2z}{\tau} I - A \right)^{-1} \right\| \leq \frac{c_0}{|z|}. \tag{2.3}$$

Passing in (2.2) to the norm and taking (2.3) into account, we obtain

$$\left\| \frac{\tau}{2} A \left(I - \frac{\tau}{2} A \right)^{k} \left(I + \frac{\tau}{2} A \right)^{-(k+1)} (I + \tau A)^{-j} \right\|$$
$$\leq \frac{c_0}{2\pi} \int_0^{+\infty} \frac{|1-z|^k}{|1+z|^{k+1} |1+2z|^j} d\rho, \tag{2.4}$$

where $z = \rho(\cos\varphi + i \sin\varphi)$.

Let us estimate the improper integral contained in the right-hand part of (2.4). It is obvious that

$$\int_0^{+\infty} \frac{|1-z|^k}{|1+z|^{k+1} |1+2z|^j} d\rho$$
$$= \int_0^{+\infty} \frac{\left(1 - 2\lambda\rho + \rho^2\right)^{\frac{k}{2}}}{\left(1 + 2\lambda\rho + \rho^2\right)^{\frac{k+1}{2}} \left(1 + 4\lambda\rho + 4\rho^2\right)^{\frac{j}{2}}} d\rho. \tag{2.5}$$

We represent integral (2.5) as a sum of three integrals

$$\int_0^{+\infty} \frac{(1 - 2\lambda\rho + \rho^2)^{\frac{k}{2}}}{(1 + 2\lambda\rho + \rho^2)^{\frac{k+1}{2}} (1 + 4\lambda\rho + 4\rho^2)^{\frac{j}{2}}} d\rho = \int_0^{2\lambda} + \int_{2\lambda}^{k+1} + \int_{k+1}^{+\infty}. \tag{2.6}$$

The validity of the following inequalities is obvious:

$$\begin{aligned} 1 + 2\lambda\rho + \rho^2 &\geq (1+\lambda\rho)^2, \\ 1 + 4\lambda\rho + 4\rho^2 &\geq (1+\lambda\rho)^2, \\ 1 - 2\lambda\rho + \rho^2 &\leq 1 \quad \text{if} \quad 0 \leq \rho \leq 2\lambda. \end{aligned}$$

Taking these inequalities into account, for the first integral from the right-hand part of equality (2.6) we obtain the estimate

$$\int_0^{2\lambda} \frac{(1 - 2\lambda\rho + \rho^2)^{\frac{k}{2}}}{(1 + 2\lambda\rho + \rho^2)^{\frac{k+1}{2}} (1 + 4\lambda\rho + 4\rho^2)^{\frac{j}{2}}} d\rho$$
$$\leq \int_0^{2\lambda} \frac{1}{(1+\lambda\rho)^{k+j+1}} d\rho \leq \int_0^{+\infty} \frac{d\rho}{(1+\lambda\rho)^{k+j+1}} \tag{2.7}$$
$$= -\frac{1}{\lambda(k+j)(1+\lambda\rho)^{k+j}} \Bigg|_0^{+\infty} = \frac{1}{\lambda(k+j)}.$$

Let us estimate the second integral in the right-hand part of equality (2.6).

For any $j \ge 1$ the following inequality is valid

$$\int_{2\lambda}^{k+1} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}} d\rho$$

$$\le \int_{2\lambda}^{k+1} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+2}{2}}} d\rho = \int_{2\lambda}^{k+1} \frac{\chi(\rho)(1-\chi(\rho))^m}{2\lambda\rho(1+\chi(\rho))^{m+1}} d\rho, \tag{2.8}$$

where

$$m = \frac{k}{2}, \quad \chi(\rho) = \frac{2\lambda\rho}{1+\rho^2}.$$

Since $0 \le \chi(\rho) < 1$ for $2\lambda \le \rho < +\infty$, we have the estimate

$$\chi(\rho)\left(1-\chi(\rho)\right)^m \le \frac{1}{m+1}\left(1-\frac{1}{m+1}\right)^m \le \frac{1}{2(m+1)}. \tag{2.9}$$

The inequality

$$\rho(1+\chi(\rho))^{m+1} \ge \rho(1+(m+1)\chi(\rho)) = \rho + 2\lambda(m+1)\frac{\rho^2}{1+\rho^2}$$

$$\ge \rho + \lambda(k+2)\frac{4\lambda^2}{1+4\lambda^2} = \rho + \lambda\lambda_0(k+2), \quad \lambda_0 = \frac{4\lambda^2}{1+4\lambda^2} \tag{2.10}$$

is also valid by virtue of the Bernoulli inequality.

Taking (2.9) and (2.10) into account, from (2.8) we obtain

$$\int_{2\lambda}^{k+1} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}} d\rho$$

$$\le \frac{1}{\lambda(k+2)} \int_{2\lambda}^{k+1} \frac{d\rho}{\rho+\lambda\lambda_0(k+2)} = \frac{1}{\lambda(k+2)} \ln\frac{k+1+\lambda\lambda_0(k+2)}{2\lambda+\lambda\lambda_0(k+2)} \tag{2.11}$$

$$\le \frac{1}{\lambda(k+2)}\ln\frac{1+\lambda\lambda_0}{\lambda\lambda_0} = \frac{1}{\lambda(k+2)}\ln\left(\frac{1}{4\lambda^3}+\frac{1}{\lambda}+1\right) \le \frac{1}{\lambda(k+2)}\ln\frac{9}{4\lambda^3}.$$

Let us show that the inequality

$$\int_{2\lambda}^{k+1} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}} d\rho$$

$$\le \frac{1}{16\lambda^3(k+j+2)} \tag{2.12}$$

is fulfilled for any $j > 2$.

By virtue of the Bernoulli inequality, the inequality

$$(1+4\lambda\rho+4\rho^2)^{\frac{j-1}{2}} \ge (1+4\rho^2)^{\frac{j-1}{2}} \ge 1+2(j-1)\rho^2$$

holds for $j>2$.

Taking this inequality into account, we obtain

$$\int_{2\lambda}^{k+1} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}}d\rho$$

$$\le \int_{2\lambda}^{k+1} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+2}{2}}(1+2(j-1)\rho^2)}d\rho$$

$$=\frac{1}{2\lambda}\int_{2\lambda}^{k+1} \frac{\chi(\rho)(1-\chi(\rho))^m}{\rho(1+\chi(\rho))^{m+1}(1+2(j-1)\rho^2)}d\rho$$

$$\le \frac{1}{\lambda(k+2)}\int_{2\lambda}^{k+1}\frac{d\rho}{\rho(1+2(j-1)\rho^2)} \le \frac{1}{\lambda(k+2)}\int_{2\lambda}^{+\infty}\frac{d\rho}{\rho(1+2(j-1)\rho^2)}$$

$$=\frac{1}{\lambda(k+2)}\int_{2\lambda}^{+\infty}\left(\frac{1}{\rho}-\frac{2(j-1)\rho}{1+2(j-1)\rho^2}\right)d\rho$$

$$=\frac{1}{\lambda(k+2)}\ln\frac{\rho}{\sqrt{1+2(j-1)\rho^2}}\Bigg|_{2\lambda}^{+\infty} = \frac{1}{\lambda(k+2)}\ln\frac{\sqrt{1+8\lambda^2(j-1)}}{2\lambda\sqrt{2(j-1)}}$$

$$=\frac{1}{2\lambda(k+2)}\ln\left(1+\frac{1}{8\lambda^2(j-1)}\right) \le \frac{1}{16\lambda^3(k+2)(j-1)}$$

$$=\frac{1}{16\lambda^3(k(j-1)+2j-2)} \le \frac{1}{16\lambda^3(2k+j+(j-2))}$$

$$\le \frac{1}{16\lambda^3(2k+j+1)} \le \frac{1}{16\lambda^3(k+k+j+1)} \le \frac{1}{16\lambda^3(k+j+2)}.$$

We have thus proved inequality (2.12).

Let us now estimate the third integral in the right-hand part of equality (2.6). Like in the case of the second integral of (2.6), we have

$$\begin{aligned}
&\int_{k+1}^{+\infty} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}}d\rho \\
&\le \int_{k+1}^{+\infty} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+2}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j-1}{2}}}d\rho \\
&\le \int_{k+1}^{+\infty}\left(\frac{\chi(\rho)(1-\chi(\rho))^k}{2\lambda\rho(1+\rho^2)(1+\chi(\rho))^{k+2}}\right)^{\frac{1}{2}}\frac{1}{(1+\rho^2)^{\frac{j-1}{2}}} \\
&\le \frac{1}{2\sqrt{\lambda(k+1)}}\int_{k+1}^{+\infty}\frac{1}{\rho^{\frac{2j+1}{2}}}d\rho = \frac{1}{\sqrt{\lambda}(2j-1)(k+1)} \le \frac{1}{\sqrt{\lambda}(k+j)}.
\end{aligned} \tag{2.13}$$

With estimates (2.7), (2.11), (2.12) and (2.13) taken into account, equality (2.6) implies the estimate

$$\int_0^{+\infty} \frac{(1-2\lambda\rho+\rho^2)^{\frac{k}{2}}}{(1+2\lambda\rho+\rho^2)^{\frac{k+1}{2}}(1+4\lambda\rho+4\rho^2)^{\frac{j}{2}}} d\rho \le \frac{c_2(\lambda)}{k+j}, \tag{2.14}$$

where

$$c_2(\lambda) = \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} + \frac{1}{\lambda\lambda_1}\right).$$

Taking (2.5) and (2.14) into account, from (2.4) we obtain integral (2.1). Lemma 2.1 is proved.

Remark 2.2. It is not difficult to observe that estimate (2.1) remains valid if the operator $(I+\tau A)^{-1}$ is replaced by the operator $\left(I+\frac{\tau}{2}A\right)^{-1}$, i.e. the following estimate is fulfilled

$$\left\| \frac{\tau}{2}A\left(I-\frac{\tau}{2}A\right)^k \left(I+\frac{\tau}{2}A\right)^{-(k+j+1)} \right\| \le \frac{c_1(\lambda)}{k+j}. \tag{2.15}$$

Remark 2.3. Using the Danford-Taylor integral , it is easy to prove the estimate

$$\left\|(\tau A)(I+\tau A)^{-k}\right\| \le \frac{c}{k}, \quad c = const > 0.$$

Theorem 2.4. *Assume that the operator A satisfies the conditions of Theorem 1.1. Then for the transition operator* $L = \left(I-\frac{\tau}{2}A\right)\left(I+\frac{\tau}{2}A\right)^{-1}$ *of the Crank-Nicolson scheme the estimate*

$$\left\|L^k\right\| \le c\ln\frac{et_k}{\tau}, \quad k = 2,\ldots,n, \tag{2.16}$$

is valid, where $\tau = T/n$ *and the constant* $c > 0$ *does not depend on* τ .

Proof. It is obvious that

$$L + I = 2S_0, \quad L - I = -\tau AS_0.$$

From these equalities it follows that

$$L^2 = -2\tau AS_0^2 + I. \tag{2.17}$$

Multiplying both parts of equality (2.17) by L^2 , we obtain

$$L^4 = -2\tau AS_0^2L^2 + L^2 = -2\tau AS_0^2L^2 + \left(-2\tau AS_0^2 + I\right) = -2\tau AS_0^2\left(L^2 + I\right) + I.$$

Further, by induction, we have

$$L^{2m} = -2\tau A S_0^2 \left(L^{2m-2} + L^{2m-4} + \ldots + I \right) + I. \tag{2.18}$$

Hence it follows that

$$L^{2m+1} = -2\tau A S_0^2 \left(L^{2m-1} + L^{2m-3} + \ldots + L \right) + L. \tag{2.19}$$

If in equality (2.18) we take the norms, use estimate (2.15) and Remark 2.3, then we obtain

$$\begin{aligned} \left\| L^{2m} \right\| &\le 2\left(\left\| \tau A L^{2m-2} S_0^2 \right\| + \left\| \tau A L^{2m-4} S_0^2 \right\| + \ldots + \left\| \tau A S_0^2 \right\| \right) + 1 \\ &\le c\left(\frac{1}{2m-1} + \frac{1}{2m-3} + \ldots + \frac{1}{2} \right) + 1 \le c \ln(2m) + 1. \end{aligned} \tag{2.20}$$

Analogously, from (2.19) we obtain

$$\left\| L^{2m+1} \right\| \le c\left(\ln\left(2m+1\right) + 1 \right). \tag{2.21}$$

(2.20) and (2.21) yield estimate (2.16).

Lemma 2.5 (see e.g. [16, ch. I]. *Let the operator A satisfy the conditions of Theorem 1.1. Then for any* $\tau > 0$ *and natural* $k\,(k \le n,\ c = T/n)$ *we have the inequality*

$$\left\| \left(I + \tau A \right)^{-k} \right\| \le c, \ \ c = const > 0.$$

Remark 2.6. For any natural k the estimate

$$\left\| (L^k - S^k) u_0 \right\| \le c_1(\lambda) \tau \left\| A u_0 \right\|, \ \ u_0 \in D(A)$$

is valid, where

$$L = \left(I - \frac{\tau}{2} A \right) \left(I + \frac{\tau}{2} A \right)^{-1}, \ \ S = \left(I + \tau A \right)^{-1}.$$

Indeed, by Lemma 2.1 the representation

$$\begin{aligned} L^k - S^k &= (L - S)\left(L^{k-1} + L^{k-2} S + \cdots + L S^{k-2} + S^{k-1} \right) \\ &= -\frac{1}{2} \tau^2 A^2 S_0 S \left(L^{k-1} + L^{k-2} S + \cdots + L S^{k-2} + S^{k-1} \right) \end{aligned}$$

implies the estimate

$$\begin{aligned} \left\| (L^k - S^k) u_0 \right\| &\le \tau \left\| A u_0 \right\| \sum_{i=1}^{k} \left\| \frac{\tau}{2} A L^{k-i} S_0 S^i \right\| \\ &\le \tau \left\| A u_0 \right\| \sum_{i=1}^{k} \frac{c_1(\lambda)}{k} = c_1(\lambda) \tau \left\| A u_0 \right\|. \end{aligned}$$

Here we have used the representation

$$L - S = \left(I - \frac{\tau}{2}A\right)\left(I + \frac{\tau}{2}A\right)^{-1} - (I + \tau A)^{-1}$$

$$= \left[\left(I - \frac{\tau}{2}A\right)(I + \tau A) - \left(I + \frac{\tau}{2}A\right)\right]\left(I + \frac{\tau}{2}A\right)^{-1}(I + \tau A)^{-1}$$

$$= -\frac{1}{2}(\tau A)^2\left(I + \frac{\tau}{2}A\right)^{-1} S.$$

Lemma 2.7. F*or any* $\tau > 0$ *and natural* k ($k \le n$, $\tau = T/n$), *the following estimate is valid*

$$\left\| SL^k \right\| \le c, \qquad c = const > 0. \tag{2.22}$$

Proof. By virtue of Lemma 2.5 and Remark 2.6 we have

$$\left\| L^k u \right\| = \left\| (L^k - S^k)u + S^k u \right\| \le \left\| (L^k - S^k)u \right\| + \left\| S^k u \right\| \le c\tau \|Au\| + \left\| S^k \right\| \|u\| \le c\left(\tau \|Au\| + \|u\|\right).$$

Taking this inequality into account, we obtain

$$\left\| SL^k u \right\| = \left\| L^k Su \right\| = \left\| L^k (I + \tau A)^{-1} u \right\| \le c\left(\tau \left\| A(I + \tau A)^{-1} u \right\| + \left\| (I + \tau A)^{-1} u \right\|\right)$$

$$\le c\left(\left\| (\tau A)(I + \tau A)^{-1} \right\| \|u\| + \left\| (I + \tau A)^{-1} \right\| \|u\|\right) \le c\|u\|.$$

Clearly, this implies (2.22).

Remark 2.8. It is obvious that, analogously to (2.22), the estimate

$$\left\| S_0 L^k \right\| \le c \tag{2.23}$$

is also valid.

3. Proof of the Basic Theorem

Let us proceed to proving the basic Theorem 1.1 (in what follows, c always denotes a positive constant).

Equation (1.1) at the point $t = t_{k-\frac{1}{2}} = \left(k - \frac{1}{2}\right)\tau$ is written in the form

$$\frac{u(t_k) - u(t_{k-1})}{\tau} + A\frac{u(t_k) + u(t_{k-1})}{2} = f\left(t_{k-\frac{1}{2}}\right) + \frac{1}{\tau}\varphi_k^{(1)} + \frac{1}{2}A\varphi_k^{(2)}, \tag{3.1}$$

where

$$\varphi_k^{(1)} = \int_{t_{k-\frac{1}{2}}}^{t_k}\int_{t_{k-\frac{1}{2}}}^{t}\left(u''(s) - u''\left(t_{k-\frac{1}{2}}\right)\right)ds\,dt + \int_{t_{k-1}}^{t_{k-\frac{1}{2}}}\int_{t}^{t_{k-\frac{1}{2}}}\left(u''\left(t_{k-\frac{1}{2}}\right) - u''(s)\right)ds\,dt,$$

$$\varphi_k^{(2)} = \int_{t_{k-\frac{1}{2}}}^{t_k}\left(u'(t) - u'\left(t_{k-\frac{1}{2}}\right)\right)dt + \int_{t_{k-1}}^{t_{k-\frac{1}{2}}}\left(u'\left(t_{k-\frac{1}{2}}\right) - u'(t)\right)dt.$$

From (1.3) and (3.1) we have

$$\frac{z_k - z_{k-1}}{\tau} + A\frac{z_k + z_{k-1}}{2} = \varphi_k, \tag{3.2}$$

where

$$z_k = u(t_k) - u_k, \qquad \varphi_k = \frac{1}{\tau}\varphi_k^{(1)} + \frac{1}{2}A\varphi_k^{(2)} + \left(f\left(t_{k-\frac{1}{2}}\right) - f_{k-\frac{1}{2}}\right).$$

From (3.2) it follows that

$$z_k = Lz_{k-1} + \tau S_0\varphi_k, \tag{3.3}$$

where

$$L = \left(I - \frac{\tau}{2}A\right)\left(I + \frac{\tau}{2}A\right)^{-1}, \quad S_0 = \left(I + \frac{\tau}{2}A\right)^{-1}.$$

From the recurrent relation (3.3) we obtain

$$z_k = L^k z_0 + \tau S_0 \sum_{i=1}^{k} L^{k-i}\varphi_i. \tag{3.4}$$

It is obvious that for φ_i the following estimate is true

$$\|\varphi_i\| \le c\tau^2 + \left\| f\left(t_{i-\frac{1}{2}}\right) - f_{i-\frac{1}{2}} \right\|. \tag{3.5}$$

From (3.4), taking into account (2.23) and also estimates (2.16) and (3.5), we obtain

$$\begin{aligned} \|z_k\| &\le \left\|L^k\right\| \|z_0\| + \tau\sum_{i=1}^{k}\left\|S_0L^{k-i}\right\| \|\varphi_i\| \\ &\le c\ln\frac{et_k}{\tau}\|z_0\| + c\tau^2 t_k + ct_k \max_{1\le i\le k}\left\| f\left(t_{i-\frac{1}{2}}\right) - f_{i-\frac{1}{2}} \right\| \end{aligned}$$

This completes the proof of Theorem 1.1.

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