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Physics

Rigorous Theoretical Arguments for Suppression of the Lamb Shift

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ABSTRACT. The main purpose is to elucidate the role of the hidden symmetry of the Dirac-Coulomb problem and to show algebraic possibilities for derivation of spectra. It is shown that the requirement of invariance of the Dirac Hamiltonian under some kind of Witten's superalgebra picks out the Coulomb potential only. It follows that the traditional view on the Coulomb potential is to be changed in the context of N=2 supersymmetry.

Keywords: *Dirac Hamiltonian, hidden symmetry, Lamb shift, Coulomb potential, Witten superalgebra, Laplace-Runge-Lentz vector.*

1. Relativistic Quantum Mechanics (Dirac Equation)

In 1916 A.Sommerfeld [1] derived the formula for the energy spectrum of hydrogen atom in relativistic mechanics using his quasiclassical quantization method. This formula looks like:

$$E = m\left\{1+\frac{(Z\alpha)^2}{\left(n-|\kappa|+\sqrt{\kappa^2-(Z\alpha)^2}\right)}\right\}^{-1/2}. \quad (1)$$

Here $|\kappa| = j+1/2$. After several years P.Dirac [2] introduced the correct relativistic equation, in which the spin of electron arises automatically. Dirac solved this equation himself for hydrogen atom with the Coulomb potential. Surprisingly enough his result coincides with Sommerfeld's one. This paradox was solved only 30-40 years later [3]. The fact is that the hydrogen atom spectrum is given by Sommerfeld formula. This formula removes all the degeneracy, which took place in NR quantum mechanics, but one degeneracy still remains – the spectrum depends only on the eigenvalues of total momentum $\vec{J} = \vec{L} + \vec{S}$ (or on its quantum number *j*). Therefore there remains a twofold degeneracy for $\kappa = \pm(j+1/2)$, according to which levels $E(S_{1/2})$ and $E(P_{1/2})$ must be degenerate, but an experimentally small shift was found by Lamb and Retherford [4]. This level shift is named as the Lamb shift. It was explained in QED [5]. It seems that one of the motivations of creation of the quantum electrodynamics was the aspiration to explain the Lamb shift [6].

The natural question arises – Is there some symmetry in the Dirac equation behind the prohibition of the Lamb shift?

Below we shall see that the answer to this question is affirmative.

Let us consider the general Dirac Hamiltonian

$$H = \vec{\alpha}\cdot\vec{p} + \beta m + V(r), \qquad (2)$$

where $V(r)$ is an arbitrary central potential, which is the 4th component of the Lorentz vector, in accordance with the minimal gauge switching. This Hamiltonian commutes with the total momentum operator $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$, where $\vec{\Sigma} = diag(\vec{\sigma},\vec{\sigma})$ is the spin matrix of fermions. It is easy to confirm, that the following operator $K = \beta\left(\vec{\Sigma}\cdot\vec{L} + 1\right)$ commutes with H. It is named as Dirac operator.

The eigenvalue of this operator is exactly k, mentioned above and the degeneracy with respect to two signs of k remains in the solution of Coulomb problem.

Let us remark that:

$\kappa = j + 1/2$, when $j = l + 1/2$, leading to levels $(S_{1/2}, P_{3/2}, etc.)$ and $\kappa = -(j+1/2)$, when $j = l - 1/2 \Rightarrow (P_{1/2}, D_{3/2}, etc.)$.

Therefore the forbidden of the Lamb shift results from $\kappa \to -\kappa$ symmetry which at the same time means the reflection of the spin direction with respect to the angular momentum direction.

Let us find the operator which reflects this sign. It is evident that such an operator, say Q_1, if it exists, must be anticommuting with K, i.e.

$$\{Q_1, K\} \equiv Q_1 K + K Q_1 = 0. \qquad (3)$$

Clearly the following operator

$$Q_2 = i\frac{Q_1 K}{\sqrt{K^2}} \qquad (4)$$

would be anticommuting both with K and Q_1. Moreover, the introduced operators have equal squares

$$\{Q_1, Q_2\} = 0, \quad Q_1^2 = Q_2^2 \equiv \tilde{H}. \qquad (5)$$

One can now construct new operators

$$Q_\pm = Q_1 \pm iQ_2.$$

They are nilpotent,

$$Q_\pm^2 = 0 \text{ and } \{Q_+, Q_-\} = 2\tilde{H}, \left[Q_i, \tilde{H}\right] = 0. \qquad (6)$$

These algebraic relations (6) define the structure, which is called Witten's algebra or N=2 superalgebra [7]. (There are anticommutators together with commutators in superalgebras. Such structures in mathematics are known as graded Lie algebras).

What happens if we require invariance of Dirac Hamiltonian with respect to this algebra? Or if we require

$$[Q_i, H] = 0,\ i = 1,2. \qquad (7)$$

Thus, we are faced with the following problem: **Find (construct) the operator(s), which anticommutes with the Dirac K operator and commutes with the Dirac Hamiltonian, H.**

Let us first construct the anticommuting operator. One of such operators is Dirac's γ^5 matrix. What else? There is a simple Theorem [8]:

If $\vec{V}$ is a vector with respect to angular momentum operator

$$\vec{L},\ i.e.\ if \left[L_i, V_j\right] = i\varepsilon_{ijk}V_k$$

and simultaneously it is perpendicular to it, $(\vec{L}\cdot\vec{V}) = (\vec{V}\cdot\vec{L}) = 0$, then the following operator $(\vec{\Sigma}\cdot\vec{V})$, which is scalar with respect to total momentum $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$, anticommutes with K:

$$\left\{K, \left(\vec{\Sigma}\cdot\vec{V}\right)\right\} = 0.$$

In general,

$$\left\{K, \hat{O}\left(\vec{\Sigma}\cdot\vec{V}\right)\right\} = 0, \qquad (8)$$

where $\hat{O}$ is commuting with K.

Armed with this theorem, one can choose physical vectors at hand, which obey the conditions of this Theorem. They are: $\vec{V} = \hat{\vec{r}}$ - Unit radius-vector, and $\vec{V} = \vec{p}$ - linear momentum vector.

There is also the Laplace-Runge-Lentz (LRL) vector for the Coulomb potential $V(r) = -\frac{\alpha}{r}$, namely

$$\vec{A} = \frac{1}{2}\left(\left[\vec{p}\times\vec{L}\right]-\left[\vec{L}\times\vec{p}\right]\right)-m\alpha\hat{\vec{r}}, \qquad (9)$$

but its inclusion is not needed, because

$$(\vec{\Sigma}\cdot\vec{A}) = \vec{\Sigma}\cdot\hat{\vec{r}} + \frac{i}{ma}\beta K(\vec{\Sigma}\cdot\vec{p}).$$

and the relevant operator is expressible through known operators, owing to the relation

$$\vec{K}\left(\vec{\Sigma}\cdot\vec{V}\right) = -i\beta\left(\dot{\Sigma}\cdot\frac{1}{2}\left[\vec{V}\times\vec{L}\right]-\left[\vec{L}\times\vec{V}\right]\right). \qquad (10)$$

Therefore we take the most general K-odd operator in the following form

$$Q_1 = x_1\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)+ix_2K\left(\vec{\Sigma}\cdot\vec{p}\right)+ix_3K\gamma^5 f(r). \qquad (11)$$

Now requiring the commutativity with the Hamiltonian,

$$[Q\ ,H] = \left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)\left\{x_2V'(r)-x_3f'(r)\right\}+$$
$$+2i\beta K\gamma^5\left\{\frac{x_1}{r}-mf(r)x_3\right\}=0,$$

it follows the relations:

$$x_2V'(r) = x_3f'(r),$$
$$x_3mf(r) = \frac{x_1}{r}.$$

Then we find

$$V(r) = \frac{x_1}{x_2}\frac{1}{mr}.$$

So, we can conclude that the only central potential for which the Dirac Hamiltonian is supersymmetric in the above sense, is the Coulomb one.

2. The Physical Meaning of the Constructed Conserved Operator

To elucidate the physical meaning of the derived operator let us make use of the obtained relations and rewrite it to more transparent form by application of Dirac's algebra:

$$Q_1 = \vec{\Sigma}\cdot\left\{\hat{\vec{r}}-\frac{i}{2ma}\beta(\left[\vec{p}\times\vec{L}\right]-\left[\vec{L}\times\vec{p}\right]\right\}+\frac{i}{mr}K\gamma^5. \qquad (12)$$

Lippmann and Johnson [9] in 1950 published this form in a brief abstract, where it was said only that this operator commutes with the Dirac Hamiltonian in Coulomb potential and replaces the LRL vector, known in NR quantum mechanics. But for an unknown reason they never published the derivation of this operator (a very curious fact in the history of 20th century physics).

For our aim it is useful to perform non-relativistic limits $\beta \to 1$, $\gamma^5 \to 0$. Then it follows

$$Q_1 \to \vec{\Sigma}\cdot\vec{A}, \qquad (13)$$

i.e. this new operator turns into the spin projection of the LRL vector.

The Lamb shift is explained in QED by taking into account the radiative corrections in the photon propagator and photon-electron vertex, which gives the following additional piece in Hamiltonian [5]:

$$\Delta V_{Lamb} \approx \frac{4\alpha^2}{3m^2}\left(\ln\frac{m}{\mu}-\frac{1}{5}\right)\delta^{(3)}(\vec{r})+ + \frac{\alpha^2}{2\pi m^2r^3}\left(\vec{\Sigma}\cdot\vec{L}\right). \qquad (14)$$

This expression does not commute with our obtained Johnson-Lippmann (JL) operator. Therefore when only Coulomb potential is considered in the Dirac equation as in the one-electron theory, Lamb shift would be always forbidden.

We see that the hidden symmetry of the Coulomb potential governs the physical phenomena in a sufficiently wide range – from planetary motion to the fine and hyperfine structure of atomic spectra.

3. Calculation of the Hydrogen Atom Spectrum

To this end let us calculate the square of obtained conserved operator by analogy to classical mechanics. This gives [8]

$$Q_1^2 = 1 + \left(\frac{K}{a}\right)^2 \left(\frac{H^2}{m^2} - 1\right). \tag{15}$$

All the operators entering here commute with each other. Therefore, one can replace them by corresponding eigenvalues and then solve from it for energy. Because of positive definiteness of Q_1^2 as the square of Hermitian operator, its minimal value is zero. This gives for the ground state energy

$$E_0 = m\left(1 - \frac{(Z\alpha)^2}{\kappa^2}\right)^{1/2}. \tag{16}$$

The full spectrum follows from this expression by using the well-known step procedure [7], which reduces in our case to the following substitution

$$\sqrt{\kappa^2 - a^2} \to \sqrt{\kappa^2 - a^2} + n - |\kappa|, \quad a \equiv Z\alpha . \tag{17}$$

Then the Sommerfeld formula follows.

Thus, in the case of Dirac equation the spectrum of the hydrogen atom is obtainable algebraically from the symmetry considerations alone.

We see that the supersymmetry requirement appears to be a very strong constraint in the framework of the Dirac Hamiltonian. While the supercharge operator, commuting with the Dirac Hamiltonian in the case of pure vector component only is intimately related to the LRL vector, but unlike the latter one relativistic supercharge participates in transformations of spin degrees of freedom. In passing to non-relativistic physics, information concerning spin–degrees of freedom disappears and hence LRL vector as a generator of algebra does not transform anything and symmetry becomes hidden as a relic of relativistic quantum mechanics.

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