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Mathematics

On the Wiener Processes in a Banach Space

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ABSTRACT. The analysis of the definition of Wiener process in a Banach space is given. It considers the definitions of generalized Wiener process and Wiener process in a weak sense. The representations of them by the sums of identically distributed independent (weakly independent) Gaussian random elements are given. *© 2013 Bull. Georg. Natl. Acad. Sci*

Key words: *Wiener processes, generalized random processes, Banach space valued random processes, Covariance operators of the random(generalized random) elements.*

Let X be a real separable Banach space, X^* - its conjugate, $\mathrm{B}(X)$ - the Borel σ-algebra of X, $(\Omega, \mathrm{B}, \mathrm{P})$ - a probability space. The continuous linear operator $T: X^* \to L_2(\Omega, \mathrm{B}, \mathrm{P})$ is called a generalized random element (GRE). We consider such a GRE, which maps X^* to a fix closed separable subspace $G \subset \mathrm{L}_2(\Omega, \mathrm{B}, \mathrm{P})$. Denote by $M_1 := L(X^*, G)$ the Banach space of the GRE with the norm $\|T\|_{M_1} := \sup_{\|x^*\| \le 1} \|Tx^*\|_{L_2}$. A random element (measurable map) $\xi : \Omega \to X$ is said to have a weak second order, if for all $x^* \in X^*$, $\mathrm{E}\left\langle \xi, x^* \right\rangle^2 < \infty$. We can realize the random element ξ as an element of M_1: $T_\xi\, x^* = \left\langle \xi, x^* \right\rangle$ (Continuity of T_ξ follows from the closed graph theorem). Denote by M_2 the linear space of all random elements of the weak second order with the norm $\|\xi\| = \|T_\xi\|$. Thus, we can assume $\mathrm{M}_2 \subset M_1$. Let $T \in M_1$. Consider the map $m_T : X^* \to R^1$, $m_T x^* = ETx^*$. This is a linear and bounded functional, therefore $m_T \in X^{**}$ and it is called the mean of GRE T. Let $T' x^* = Tx^* - \left\langle m_T, x^* \right\rangle$. The covariance operator of the GRE T is called the operator $R_T = T'^* T'$. $R_T : X^* \to X^{**}$ is a positive and symmetric linear operator. Further, without the loss of generality, we consider random elements (GRE) with the mean 0. If $T = T_\xi \in M_2$, then R_T maps X^* to X (see [1], theorem 3.2.1). If R is a positive and symmetric linear operator from X^* to X, then there exist $(x^*_k)_{k \in N} \subset X^*$ and

$(x_k)_{k\in N}\subset X$ such, that $\langle Rx_k^*, x_j^*\rangle=\delta_{kj}$, $Rx_k^*=x_k$, $Rx^*=\sum_{k=1}^{\infty}\langle x_k, x^*\rangle x_k$, $x^*\in X^*$ (see [1] lemma 3.1.1).

In general, as G is a separable, for $R: X^*\to X^{**}$ there exist $(x^*_k)_{k\in N}\subset X^*$ and $(x^{**}_k)_{k\in N}\subset X^{**}$ such that $\langle Rx_k^*, x_j^*\rangle=\delta_{kj}$ $Rx_k^*=x_k^{**}$, $Rx^*=\sum_{k=1}^{\infty}\langle x^{**}_k, x^*\rangle x^{**}_k$, $x^*\in X^*$.

We consider weakly independent random elements in a Banach space.

Definition 1. Random elements $\xi_1,\xi_2,\ldots,\xi_n$ are called weakly independent in X ([1: 259; 5]), if for all $x^*\in X^*$, the random variables $\langle\xi_1, x^*\rangle,\langle\xi_2, x^*\rangle,\ldots,\langle\xi_n, x^*\rangle$ are independent.

Proposition 1. *If the weak second order random elements $\xi_1,\xi_2,\ldots,\xi_n$ are weakly independent, then the cross-covariance operators of the random elements ξ_i and ξ_j $i,j=1,\ldots n$ are antisymmetric; the Gaussian random elements $\xi_1,\xi_2,\ldots,\xi_n$ are weakly independent if and only if their cross-covariance operators are antisymmetric.*

Proof. As ξ_i and ξ_j are weakly independent, for all x^* and y^* from X^*, $E\langle\xi_i, x^*+y^*\rangle\ \xi_j, x^*+y^*\rangle=0$, but $E\langle\xi_i, x^*+y^*\rangle\langle\xi_j, x^*+y^*\rangle=E\langle\xi_i, x^*\rangle\langle\xi_j, x^*\rangle+E\langle\xi_i, x^*\rangle\langle\xi_j, y^*\rangle+E\langle\xi_i, y^*\rangle\langle\xi_j, x^*\rangle+E\langle\xi_i, y^*\rangle\langle\xi_j, y^*\rangle=$ $=\langle\xi_i, x^*\rangle\xi_j, y^*\rangle+E\langle\xi_i, y^*\rangle\langle\xi_j, x^*\rangle=0$. Therefore, $E\langle\xi_i, x^*\rangle\langle\xi_j, y^*\rangle=-E\langle\xi_i, y^*\rangle\ \langle\xi_j, x^*\rangle$.

That is, $\langle R_{ij}x^*, y^*\rangle=-\langle R_{ij}y^*, x^*\rangle$. If $\xi_1,\xi_2,\ldots,\xi_n$ are Gaussian and $\langle R_{ij}x^*, y^*\rangle=-\langle R_{ij}y^*, x^*\rangle$, then the random variables $\langle\xi_1, x^*\rangle,\langle\xi_2, x^*\rangle,\ldots,\langle\xi_n, x^*\rangle$ are non-correlated, therefore, they are independent, that is, the random elements $\xi_1,\xi_2,\ldots,\xi_n$ are Gaussian weakly independent. Consequently, if the random elements $\xi_1,\xi_2,\cdots\xi_n$ are weakly independent, then the covariance operator of X^n valued random element $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$ maps $(X^*)^n$ to X^n and is given by

$$R=\begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ -R_{12} & R_{22} & \cdots & R_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ -R_{1n} & -R_{2n} & \cdots & R_{nn} \end{pmatrix},$$

where $R_{ij}^*=-R_{ij}=R_{ji}$. Now we consider the weakly independent GREs.

Definition 2. The generalized random elements $T_1, T_2,\ldots T_n$ are called weakly independent, if for all $x^*\in X^*$, the random variables $T_1x^*, T_2x^*,\ldots T_nx^*$ are independent.

If the GREs $T_1, T_2,\ldots T_n$ are weakly independent, then the cross-covariance operators of the GREs T_i and T_j $i,j=1,\ldots n$ are antisymmetric: $R_{ij}: X^*\to X^{**}$, $\langle R_{ij}x^*, y^*\rangle=E\langle T_ix^*T_jy^*=-\langle R_{ij}y^*, x^*\rangle=-ET_iy^*T_jx^*$. Gaussian GREs are weakly independent if, and only if, their cross-covariance operators are anti-symmetric. The following proposition gives the representation of the GRE by the sum of non-correlated random variables. If the GRE is Gaussian, then, obviously, the corresponding random variables are independent standard Gaussian.

Proposition 2. *Let T be a GRE. There exist* $(x_k^*)_{k\in N}$ *and* $(x^{**}{}_k)_{k\in N}\subset X^{**}$ *such that for all* $x^*\in X^*$,

$Tx^*=\sum_{k=1}^{\infty}\langle x^*,x_k^{**}\rangle Tx_k^*$, $\langle R_T x_k^*,x_j^*\rangle = ETx_k^*Tx_j^*=\delta_{kj}$, $R_T x_k^*=x_k^{**}$, $R_T x^*=\sum_{k=1}^{\infty}\langle x^{**}{}_k,x^*\rangle x^{**}{}_k$. *Therefore, if T is Gaussian, then* Tx_k, $k=1,2,...$ *are independent standard Gaussian random variables.*

Proof. Consider the covariance operator of the GRE T, $R_T : X^*\to X^{**}$, $R_T=T^*T$. Let $(x_k^*)_{k\in N}$ and $(x^{**}{}_k)_{k\in N}\subset X^{**}$ be such that $\langle R_T x_k^*,x_j^*\rangle = ETx_k^*Tx_j^*=\delta_{kj}$, $R_T x_k^*=x_k^{**}$ $R_T x^*=\sum_{k=1}^{\infty}\langle x^{**}{}_k,x^*\rangle x^{**}{}_k$, for all $x^*\in X^*$. If we take up $T_n x^*=\sum_{k=1}^{n}\langle x^*,x_k^{**}\rangle Tx_k^*$, then $E(Tx^*-T_nx^*)^2=ETx^{*2}-2ETx^*T_nx^*+T_nx^{*2}=$

$=\sum_{k=1}^{\infty}\langle x_k^{**},x^*\rangle^2-2\sum_{k=1}^{n}\langle x_k^{**},x^*\rangle^2+\sum_{k=1}^{n}\langle x_k^{**},x^*\rangle^2=\sum_{k=n+1}^{\infty}\langle x_k^{**},x^*\rangle^2\to 0$. Therefore $Tx^*=\sum_{k=1}^{\infty}\langle x^*,x_k^{**}\rangle Tx_k^*$.

The family of the GRE $(T_t)_{t\in[0,1]}$ is called a generalized random process (GRP).

Definition 3. A GRP $(T_t)_{t\in[0,1]}$ is called Gaussian, if for any natural number n, $t_1,t_2,...,t_n$ from [0,1], $x^*{}_1,x^*{}_2,...,x^*{}_n$ from X^*, the random vector $(T_1x_1^*,T_2x_2^*,...,T_nx_n^*)$ is Gaussian.

Definition 4. A Gaussian GRP $(T_t)_{t\in[0,1]}$ is called generalized Wiener process in a weak sense, if for all $x^*\in X^*$, T_tx^* is a Wiener process. The variance $ET_tx^*T_sx^*=\langle Rx^*,x^*\rangle\min(t,s)$, where $t,s,\in[0,1]$ and $R:X^*\to X^{**}$ is the covariance operator of the GRE T_1; $R=T_1^*T_1$.

Proposition 3. *Let* $(T_t)_{t\in[0,1]}$ *be a generalized Wiener process in a weak sense. There exist* $(x_k^*)_{k\in N}\subset X^*$, $(x^{**}{}_k)_{k\in N}\subset X^{**}$ *and a sequence of real valued standard Wiener processes* $(w_k(t))_{k\in N}$ *such that, for all* $k\neq j$, *and fix* $t\in[0,1]$, $w_k(t)$ *and* $w_j(t)$ *are independent, for all* $x^*\in X^*$,

$T_tx^*=\sum_{k=1}^{\infty}\langle x^*,x_k^{**}\rangle w_k(t)$, $R_1x_k^*=x_k^{**}$, $R_1x^*=\sum_{k=1}^{\infty}\langle x^{**}{}_k,x^*\rangle x^{**}{}_k$.

Proof. Consider the covariance operator of the GRE T_1, $R_1 : X^*\to X^{**}$, $R_1=T_1^*T_1$. Let $(x_k^*)_{k\in N}$ and $(x^{**}{}_k)_{k\in N}\subset X^{**}$ be such that $\langle R_1x_k^*,x_j^*\rangle=ET_1x_k^*T_1x_j^*=\delta_{kj}$, $R_1x_k^*=x_k^{**}$, $R_1x^*=\sum_{k=1}^{\infty}\langle x^{**}{}_k,x^*\rangle x^{**}{}_k$, for all $x^*\in X^*$. Denote the real valued processes $(w_k(t))_{t\in[0,1]}:=(T_tx_k^*)_{t\in[0,1]}$, $k=1,2,...$

As $\langle R_1x_k^*,x_j^*\rangle=ET_1x_k^*T_1x_j^*=\delta_{kj}$, $w_k(t)$ and $w_j(t)$ are independent. Then

$$E(T_tx^*-\sum_{k=1}^{n}\langle x^*,x_k^{**}\rangle w_k(t))^2=ET_tx^{*2}-2t\sum_{k=1}^{n}\langle x^*,x_k^{**}\rangle^2+t\sum_{k=1}^{n}\langle x^*,x_k^{**}\rangle^2=$$

$$=t\sum_{k=n+1}^{\infty}\langle x^*,x_k^{**}\rangle^2\to 0\text{. Therefore }T_tx^*=\sum_{k=1}^{\infty}\langle x^*,x_k^{**}\rangle w_k(t).$$

Remark. From the definition of the generalized Wiener process in a weak sense $(T_t)_{t\in[0,1]}$, as $ET_t(x_k^*+x_j^*)T_s(x_k^*+x_j^*)=\min(t,s)\langle R(x_k^*+x_j^*),(x_k^*+x_j^*)\rangle = 2\min(t,s)$, it follows that for the Wiener processes $(w_k(t))_{k\in N}$ in the representation $T_t x^* = \sum_{k=1}^{\infty}\langle x^*, x_k^{**}\rangle w_k(t)$, $Ew_k(t)w_j(s) = -Ew_k(s)w_j(t)$. That is, the random processes $w_k(t)$ and $w_j(t)$ as random elements in C[0,1] are weakly independent.

Now we introduce a very important and well-known definition of a white noise.

Definition 5. Let *X* be a real separable Hilbert space *H*. A Gaussian GRE in *H* with covariance operator *R=I* ($I: H\to H$ is an identical operator) is called a white noise.

Remark. If $(e_k)_{k\in N}$ is an orthonormal basis in *H* and $(\gamma_k)_{k\in N}$ is the sequence of independent standard Gaussian random variables (identically distributed with mean 0 and variance 1), then the sum $\sum_{k=1}^{\infty} e_k\gamma_k$, which does not converge in *H*, represents a white noise $T: H\to G$, $Th=\sum_{k=1}^{\infty}\langle e_k,h\rangle\gamma_k$. Conversely, if we have a white noise $T: H\to G$, then the random variables Te_k, $k\in N$ are standard Gaussian and orthogonal as $ETe_kTe_j = \langle Ie_k, e_j\rangle = \langle e_k,e_j\rangle = \delta_{kj}$, that is, they are independent. Denote $Te_k=\gamma_k$, then

$$Th = T\sum_{k=1}^{\infty}\langle e_k,h\rangle e_k = \sum_{k=1}^{\infty}\langle e_k,h\rangle Te_k = \sum_{k=1}^{\infty}\langle e_k,h\rangle\gamma_k .$$

Definition 6. Let *H* be a separable Hilbert space. A Gaussian GRP $(\Upsilon_t)_{t\in[0,1]}$ is called the canonical generalized Wiener process in a weak sense in H, if, for all $h\in H$, $\Upsilon_t h$ is a Wiener process and the variance $E\Upsilon_t h\Upsilon_s h = \min(t,s)\langle h,h\rangle$, $t,s\in[0,1]$ and $h\in H$. That is, the covariance operator of the GRE Υ_1 is an identical operator $I: H\to H$, that means, Υ_1 is a white noise in *H*.

Remark. For any sequence of real valued Wiener processes $(w_k(t))_{k\in N}$, such that for all $k\neq j$ and fix $t\in[0,1]$, $w_k(t)$ and $w_j(t)$ are independent and for any orthonormal basis $(e_k)_{k\in N}$ in a separable Hilbert space *H*, the sum $\Upsilon_t = \sum_{k=1}^{\infty} e_k w_k(t)$ is a canonical generalized Wiener process in a weak sense in *H*. Indeed, consider the GRP $T_t h = \Upsilon_t h = \sum_{k=1}^{\infty}\langle e_k,h\rangle w_k(t)$. It is easy to see that $T_t h$ is a Wiener process and $E\Upsilon_t h\Upsilon_s h = \min(t,s)\langle h,h\rangle$, therefore, $(\Upsilon_t)_{t\in[0,1]}$ is a canonical generalized Wiener process in a weak sense.

Now we show that every generalized Wiener process in a weak sense in a separable Banach space *X* is an image of a canonical generalized Wiener process in a weak sense by the linear bounded operator .

Proposition 4. *For any generalized Wiener process* $(T_t)_{t\in[0,1]}$ *in a weak sense in a separable Banach space X, there exist a separable Hilbert space H, a linear, bounded operator* $A: X^*\to H$ *and a canonical generalized Wiener process* $(\Upsilon_t)_{t\in[0,1]}$ *in a weak sense in H such that* $T_t = A^*\Upsilon_t = \sum_{k=1}^{\infty} A^* e_k w_k(t)$ *and* $R = T_1^* T_1 = A^* A$.

Proof. Let $(T_t)_{t\in[0,1]}$ be a generalized Wiener process in X. By the proposition 2 there exist $(x_k^*)_{k\in N} \subset X^*$, $(x^{**}_k)_{k\in N} \subset X^{**}$ and a sequence of independent real valued standard Wiener processes $(w_k(t))_{k\in N}$ such that for all $x^* \in X^*$, $T_t x^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**}\rangle w_k(t)$, $R_1 x_k^* = x_k^{**}$, $\langle R_1 x_k^*, x_j^*\rangle = ET_1 x_k^* T_1 x_j^* = \delta_{kj}$, $R_1 x^* = \sum_{k=1}^{\infty} \langle x^{**}_k, x^*\rangle x^{**}_k$. By the factorization lemma (see [1], lemma 3.1.1), there exist a separable Hilbert space H and a linear bounded operator $A: X^* \to H$ such that $R_1 = A^* A$. Then $\langle R_1 x_k^*, x_j^*\rangle =$ $= \langle A^* A x_k^*, x_j^*\rangle = \langle A x_k^*, A x_j^*\rangle = \delta_{kj}$ $k, j = 1, 2, \ldots$ Therefore $e_k := A x_k^* \in H$ $k = 1, 2, \ldots$, is an orthonormal sequence and $T_t x^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**}\rangle w_k(t) = \sum_{k=1}^{\infty} \langle R_1 x_k^*, x^*\rangle w_k(t) = \sum_{k=1}^{\infty} \langle A^* A x_k^*, x^*\rangle = \sum_{k=1}^{\infty} \langle A^* e_k, x^*\rangle w_k(t)$. For this sense we write $T_t = A^* \Upsilon_t = \sum_{k=1}^{\infty} A^* e_k w_k(t)$.

Definition 7. A Gaussian GRP $(T_t)_{t\in[0,1]}$ is called a generalized Wiener process if, for all $x^* \in X^*$, $T_t x^*$ is a Wiener process and for all $t, s, \in [0,1]$ and $x^*, y^* \in X^*$, the variance $ET_t x^* T_s y^* = \langle Rx^*, y^*\rangle \min(t,s)$, where $R: X^* \to X^{**}$ is the covariance operator of the GRE T_1; $R = T_1^* T_1$.

Proposition 5. *Let* $(T_t)_{t\in[0,1]}$ *be a generalized Wiener process. There exist* $(x_k^*)_{k\in N} \subset X^*$, $(x^{**}_k)_{k\in N} \subset X^{**}$ *and a sequence of real valued independent standard Wiener processes* $(w_k(t))_{k\in N}$ *such that for all* $x^* \in X^*$, $T_t x^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**}\rangle w_k(t)$, $R_1 x_k^* = x_k^{**}$, $R_1 x^* = \sum_{k=1}^{\infty} \langle x^{**}_k, x^*\rangle x^{**}_k$.

Proof. As a generalized Wiener process is the generalized Wiener process in a weak sense, by the proposition 3, there exist $(x_k^*)_{k\in N} \subset X^*$, $(x^{**}_k)_{k\in N} \subset X^{**}$ and a sequence of real valued standard Wiener processes $(w_k(t))_{k\in N}$ such that, for all $k \neq j$, and fix $t \in [0,1]$, $w_k(t)$ and $w_j(t)$ are independent, for all $x^* \in X^*$, $T_t x^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**}\rangle w_k(t)$, $R_1 x_k^* = x_k^{**}$, $R_1 x^* = \sum_{k=1}^{\infty} \langle x^{**}_k, x^*\rangle x^{**}_k$. From the condition $ET_t x^* T_s y^* = \langle Rx^*, y^*\rangle \min(t,s)$ it follows, that $Ew_k(t) w_j(s) = \langle Rx_k^*, x_j^*\rangle \min(t,s) = 0$, that is w_k and w_j are independent for all $k \neq j$.

Definition 8. Let H be a separable Hilbert space. A Gaussian GRP $(\Upsilon'_t)_{t\in[0,1]}$ is called a canonical generalized Wiener process in H, if, for all $h \in H$, $\Upsilon'_t h$ is a Wiener process and the variance $E\Upsilon'_t h \Upsilon'_s l = \min(t,s)\langle h, l\rangle$, $t, s \in [0,1]$ and $h, l \in H$. That is, the covariance operator of the GRE Υ_1' is an identical operator $I: H \to H$, that means Υ_1' is a white noise in H.

Remark. For any sequence of real valued independent standard Wiener processes $(w_k(t))_{k\in N}$ and an orthonormal basis $(e_k)_{k\in N}$ in a separable Hilbert space *H*, the sum $\Upsilon'_t = \sum_{k=1}^{\infty} e_k w_k(t)$ is a canonical generalized Wiener process in *H*. Indeed, consider the GRP $T_t h = \Upsilon'_t h = \sum_{k=1}^{\infty} \langle e_k, h\rangle w_k(t)$. It is easy to see that $T_t h$ is a Wiener process and $E\Upsilon'_t h \Upsilon'_s l = \min(t,s)\langle h,l\rangle$, therefore, $(\Upsilon'_t)_{t\in[0,1]}$ is a canonical generalized Wiener process.

Proposition 6. *For any generalized Wiener process $(T_t)_{t\in[0,1]}$ in a separable Banach space X, there exist a separable Hilbert space H, a linear bounded Operator $A: X^* \to H$ and a canonical generalized Wiener process $(\Upsilon'_t)_{t\in[0,1]}$ in H such that $T_t = A^*\Upsilon'_t = \sum_{k=1}^{\infty} A^* e_k w_k(t)$ and $R = T_1^* T_1 = A^* A$.*

Proof. Let $(T_t)_{t\in[0,1]}$ be a generalized Wiener process in *X*. By the proposition 5, there exist $(x_k^*)_{k\in N} \subset X^*$, $(x^{**}_k)_{k\in N} \subset X^{**}$ and a sequence of real-valued independent standard Wiener processes $(w_k(t))_{k\in N}$ such that for all $x^* \in X^*$, $T_t x^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**}\rangle w_k(t)$, $R_1 x_k^* = x_k^{**}$, $R_1 x^* = \sum_{k=1}^{\infty} \langle x^{**}_k, x^*\rangle x^{**}_k$. By the factorization lemma ([1, lemma 3.1.1]), there exist a separable Hilbert space *H* and a linear bounded operator $A: X^* \to H$ such that $R_1 = A^* A$. Then $\langle R_1 x_k^*, x_j^*\rangle = \langle A^* A x_k^*, x_j^*\rangle = \langle A x_k^*, A x_j^*\rangle = \delta_{kj}$ $k, j, = 1, 2, \ldots$ Therefore, $e_k := A x_k^* \in H$, $k = 1, 2, \ldots$ is an orthonormal sequence and $T_t x^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**}\rangle w_k(t) =$

$= \sum_{k=1}^{\infty} \langle R_1 x_k^*, x^*\rangle w_k(t) = \sum_{k=1}^{\infty} \langle A^* A x_k^*, x^*\rangle = \sum_{k=1}^{\infty} \langle A^* e_k, x^*\rangle w_k(t)$. For this sense we write $T_t = A^* \Upsilon'_t = \sum_{k=1}^{\infty} A^* e_k w_k(t)$.

Consider now a Wiener process in a separable Banach space.

Definition 9. A family of random elements (random process) $(W(t))_{t\in[0,1]}$ is called a Wiener process in a separable Banach space *X*, if W(0)=0 a.s.; for any $0 \le t_0 < t_1 < \ldots < t_n \le 1$, the random elements $W(t_{i+1}) - W(t_i)$, $i = 0,1 \ldots n-1$ are independent; for any $t \in [0,1]$, $W(t)$ is a Gaussian random element with a mean 0 and a covariance operator tR, where $R: X^* \to X$ is a Gaussian covariance; $(W(t))_{t\in[0,1]}$ has continuous sample paths.

Description of the class of Gaussian covariance operators is a very important problem (see [1]). For example, in the Hilbert space case, the operator $R: H \to H$ is a Gaussian covariance if, and only if, *R* is a nuclear operator.

Proposition 7. *The generalized Wiener process $(T_t)_{t\in[0,1]}$ generates a Wiener process in a separable Banach space X if, and only if, the covariance operator $R = T_1^* T_1$ maps X^* to X and is a Gaussian covariance. That is, there exists the Wiener process $(W(t))_{t\in[0,1]}$ such that for all $x^* \in X^*$, $T_t x^* = \langle W(t), x^*\rangle$ a.s.*

Proof. Let $(T_t)_{t\in[0,1]}$ be a generalized Wiener process and $R = T_1^* T_1 : X^* \to X$ be a Gaussian covariance, then there exist $(x^*_k)_{k\in N} \subset X^*$ and $(x_k)_{k\in N} \subset X$ such that $\langle Rx_k^*, x_j^* \rangle = \delta_{kj}$, $Rx_k^* = x_k$, $Rx^* = \sum_{k=1}^{\infty} \langle x_k, x^* \rangle x_k$, $x^* \in X^*$. Consider the GRP $T_t x^* = \sum_{k=1}^{\infty} \langle x^*, x_k \rangle w_k(t)$. As R is a Gaussian covariance, by the Ito-Nisio theorem (see[2]), we have convergence of the sum $\sum_{k=1}^{\infty} x_k w_k(t)$ in the Banach space X. Denote $W(t) = \sum_{k=1}^{\infty} x_k w_k(t)$. It is easy to see that $(W(t))_{t\in[0,1]}$ is a Wiener process in a Banach space X. Conversely, if $(W(t))_{t\in[0,1]}$ is a Wiener process in X, then $T_t x^* = \langle W(t), x^* \rangle$, $x^* \in X^*$, $t \in [0,1]$ is a generalized Wiener process.

Proposition 8. $(W(t))_{t\in[0,1]}$ *is a Wiener process in a Banach space X, if, and only if, there exist a separable Hilbert space, a canonical generalized Wiener process* $(\Upsilon'_t)_{t\in[0,1]}$ *in it and a linear bounded operator* $A : X^* \to H$, *such that* $A^* A$ *is a Gaussian covariance and* $W(t) = A^* \Upsilon'_t = \sum_{k=1}^{\infty} A^* e_k w_k(t)$. *The last sum converges in X a.s.*

Proof. Let $(W(t))_{t\in[0,1]}$ be a Wiener process in X, then $Rx^* = R_{W_1} x^* = \sum_{k=1}^{\infty} \langle x_k, x^* \rangle x_k =$

$= \sum_{k=1}^{\infty} \langle A^* A x_k^*, x^* \rangle A^* A x_k^* = \sum_{k=1}^{\infty} \langle A^* e_k, x^\bullet \rangle A^* e_k$, where $A : X^* \to H$, H is a separable Hilbert space and $(e_k)_{k\in N}$ is an orthonormal basis on it. By the propositions 4 and 5, we have $W(t) = A^* \sum_{k=1}^{\infty} e_k w_k(t) =$

$= \sum_{k=1}^{\infty} A^* e_k w_k(t)$ and the last sum converges a.s. in X.

It is true more deep results (see [3] and [4]), which show that the convergence of the sum in proposition 7 is uniform for t a.s. (for $\omega \in \Omega$) and there exists another representation of the Wiener process in a Banach space by the sum of independent, identically distributed Gaussian random elements uniformly converging for t a.s. (for $\omega \in \Omega$).

Let us now define Wiener processes in a weak sense.

Definition 10. A family of Gaussian random elements is called a Wiener process in a weak sense in a separable Banach space X, if, for all $x^* \in X^*$, $(\langle W_t, x^* \rangle)_{t\in[0,1]}$ is a real valued Wiener process with variance $t\langle Rx^*, x^* \rangle$, where R is the covariance operator of the random element W_1.

Remark. It is clear that Wiener process in a common sense is a Wiener process in a weak sense. We give in R^2 a simple example of a Wiener process in a weak sense, which isn't a Wiener process in a common sense.

Example 1. Let $e_k, k=0,1,...$ be a Haar orthonormal basis in $L_2[0,1]$. It is easy to see, that if $t=1$, then $\int_0^1 e_0(\tau)d\tau = 1$ and $\int_0^1 e_k(\tau)d\tau = 0$ for all $k \geq 1$; if $t=\frac{1}{2}$, then $\int_0^{1/2} e_0(\tau)d\tau = \frac{1}{2}$, $\int_0^{1/2} e_1(\tau)d\tau = \frac{1}{2}$, and $\int_0^{1/2} e_k(\tau)d\tau = 0$ for all $k \geq 2$ and so forth, if $t_k = \frac{2^k - 2^{n+1}+1}{2^{n+1}}$, for any natural n and $2^n \leq k < 2^{n+1}$, we have $\int_0^{t_k} e_k(\tau)d\tau \neq 0$ and for all $m > k$ $\int_0^{t_k} e_m(\tau)d\tau = 0$.

Let R be a 4×4 dimension matrix

$$R = \begin{pmatrix} \sigma & 0 & 0 & \alpha \\ 0 & \sigma & -\alpha & 0 \\ 0 & -\alpha & \sigma & 0 \\ \alpha & 0 & 0 & \sigma \end{pmatrix}, (\sigma > \alpha).$$

Let $(\vec{\eta}_n)_{n\in N} := (\eta_1{}^{(n)}, \eta_2{}^{(n)}, \eta_3{}^{(n)}, \eta_4{}^{(n)})_{n\in N}$ be a sequence of independent, identically distributed Gaussian random vectors in R^4 with a mean 0 and a covariance operator R. Then the random elements $(\vec{\xi}_n)_{n\in N}$ in R^2 $\vec{\xi}_{2n} = (\eta_1{}^{(n)}, \eta_2{}^{(n)})$ and $\vec{\xi}_{2n+1} = (\eta_3{}^{(n)}, \eta_4{}^{(n)})$, $n=0,1,...$ are weakly independent Gaussian random elements in R^2. If we consider the R^2-valued random process $W(t) = \sum_{k=1}^{\infty} \int_0^t e_k(\tau)d\tau \xi_k$, where $(e_k)_{k\in N}$ in $L_2[0,1]$ is a Haar basis, then it is easy to see that $(W(t))_{t\in[0,1]}$ is a Wiener process in a weak sense, but it isn't a Wiener process in a common sense in R^2; indeed, we show for example, that $W(\frac{1}{2})$ and $W(1)-W(\frac{1}{2})$ are not independent. It is easy to see that $W_1 = \int_0^1 e_0(\tau)d\tau\vec{\xi}_0 = \xi_0$ and $W_{\frac{1}{2}} = \int_0^{1/2} e_0(\tau)d\tau\vec{\xi}_0 + \int_0^{1/2} e_1(\tau)d\tau\vec{\xi}_1 = \frac{1}{2}(\xi_0 + \xi_1)$.

Let $f, g \in R^2$ be such that $E\langle \xi_1, f\rangle\langle \xi_0, g\rangle \neq o$

$$E\langle W_{1/2}, f\rangle\langle W_1 - W_{1/2}, g\rangle = \tfrac{1}{4}E\langle \xi_0+\xi_1, f\rangle\langle \xi_0 - \xi_1, g\rangle = \tfrac{1}{2}E\langle \xi_1, f\rangle\langle \xi_0, g\rangle \neq 0.$$

Therefore, $W(\frac{1}{2})$ and $W(1)-W(\frac{1}{2})$ are not independent.

Now we provide representations of the Wiener process in a weak sense with the sum of weakly independent identically distributed Gaussian random elements in X. Remember that the random elements $\xi_1, \xi_2, ..., \xi_n$ are called weakly independent, if, for all $x^* \in X^*$, the random variables $\langle \xi_1, x^*\rangle, \langle \xi_2, x^*\rangle, ..., \langle \xi_n, x^*\rangle$ are independent.

The following theorem is true.

Theorem 1. *Let* $(e_k)_{k\in N}$ *be a Haar orthonormal basis in* $L_2[0,1]$, $\xi_1,\xi_2,....$ *be a sequence of weakly independent identically distributed Gaussian random elements in X, then the sum* $\sum_{k=1}^{\infty}\int_0^t e_k(\tau)d\tau\xi_k$ *converges uniformly for t a.s. (for* $\omega\in\Omega$*) in X to the Wiener process in a weak sense.*

The proof of this theorem is analogous to the proof of the theorem 1 in the case when $(e_k)_{k\in N}$ is a Haar orthonormal basis ([6, 126]).

Now we show that every Wiener process in a weak sense can be represented as a sum from Theorem 1.

Theorem 2. *Let* $(W(t))_{t\in[0,1]}$ *be a Wiener process in a weak sense with the covariance operator R of the random element* $W(1)$*, there exists the sequence of weakly independent, identically distributed Gaussian random elements* $(\xi_k)_{k\in N}$ *with a mean 0 and a covariance operator R such that* $W(t)=\sum_{k=1}^{\infty}\int_0^t e_k(\tau)d\tau\xi_k$ *a.s., where* $(e_k)_{k\in N}$ *is a Haar orthonormal basis in* $L_2[0,1]$.

Proof. For any fix $x^*\in X$, the random process $(\langle W_t,x^*\rangle)_{t\in[0,1]}$ is a real valued Wiener process. Therefore, there exists the sequence $(\gamma_k((x^*)))_{k\in N}$ of standard Gaussian independent random variables such that $\langle W(t),x^*\rangle=\langle Rx^*,x^*\rangle\sum_{k=1}^{\infty}\int_0^t e_k(\tau)d\tau\gamma_k(x^*)$. For any fix $k\in N$ consider the GRE $T_k:X^*\to G$, $T_kx^*=\langle Rx^*,x^*\rangle^{1/2}\gamma_k(x^*)$. It is easy to see that this definition is correct. *T* is a Gaussian GRE with the covariance operator *R*. As *R* is a Gaussian covariance, there exists the Gaussian random element ξ_k in *X* with a mean 0 and a covariance operator *R* such that $\langle\xi_k,x^*\rangle=T_kx^*=\langle Rx^*,x^*\rangle^{1/2}\gamma_k(x^*)$, $x^*\in X$. It is clear that the random elements ξ_k, $k=1,2,...$ are weakly independent. Therefore, we have $\sum_{k=1}^{\infty}\int_0^t e_k(\tau)d\tau\xi_k$.

From theorems 2 and 3 follows

Corollary. *Wiener process in a weak sense has a.s. continuous sample paths.*

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მათემატიკა

ვინერის პროცესები ბანახის სივრცეში

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