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Physics

Classical Motion of a Test Particle in the Bonnor Spacetime Based on Lyra Manifold

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ABSTRACT. The exact solutions of the vacuum field equations for the Bonnor spacetime in presence of a massless scalar field within the framework of the Lyra manifold are studied. Also, the classical motion of a test particle in this spacetime by using the Hamilton-Jacobi method is investigated. *© 2013 Bull. Georg. Natl. Acad. Sci.*

Key words: *Bonnor spacetime, Lyra manifold, particle trajectory.*

1. Introduction

Einstein provided a general theory of gravitation by geometry and this theory has been very successful in describing the gravitational phenomena. Einstein field equations without the cosmological constant admitted only nonstatic solutions and he introduced the cosmological constant in order to obtain the static models. The properties of the spacetime require the Riemannian geometry for their description. Several modifications of Riemannian geometry have been suggested to unify gravitation, electromagnetism and other effects in universe. One of the modified theories has been introduced by Lyra [1]. He introduced an additional gauge function into the structureless manifold as a result of which a displacement vector field arises naturally from the geometry. The Einstein field equations in normal gauge based on Lyra manifold defined by Sen [2] and Sen and Dunn [3] as

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \frac{3}{4}\left(2\xi_\alpha \xi_\beta - g_{\alpha\beta}\xi_\mu \xi^\mu\right) = T_{\alpha\beta}, \tag{1}$$

where ξ_α is the Lyra displacement vector field, other symbols have their usual meaning as in Riemannian geometry and we choose the geometric units in which $8\pi G = c = 1$. In Lyra formalism, the constant displacement vector field plays the same role as the cosmological constant in the standard general relativity [4]. Also, the scalar-tensor treatment based on Lyra manifold predicts some effects, within the observational limit, as in

Einstein theory [4]. In this paper, we will determine the parameters of motion for a moving test particle in the Bonnor spacetime based on Lyra manifold.

2. The metric and field equations

We assume that the metric of the spacetime is of the Bonnor form, in the cylindrical coordinates, with the following line element [5]:

$$ds^2 = r^{2m^2} G^2 (dr^2 - dt^2) + r^2 G^2 d\phi^2 + G^{-2} dz^2, \tag{2}$$

here G is an unknown function of r and m is a constant. In this analysis, we assume that the energy-momentum tensor corresponding to the massless scalar field Φ is defined as

$$T_{\mu\nu} = \Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2} g_{\mu\nu}\Phi_{,\sigma}\Phi^{,\sigma}, \tag{3}$$

the comma denotes partial derivatives with respect to the appropriate coordinates and Φ satisfies the Klein-Gordon equation as

$$\Box\Phi = 0, \tag{4}$$

where $\Box\Phi = \frac{1}{\sqrt{-g}}\left[\sqrt{-g}\,g^{\mu\nu}\Phi_{,\nu}\right]_{,\mu}$ while $g = \det(g_{\mu\nu})$. By considering $\Phi = \Phi(r,t)$, the equation (4) is changed to the following relation

$$\ddot{\Phi} - \Phi'' - \frac{1}{r}\Phi' = 0, \tag{5}$$

where the over head dot and prime indicate partial differentiation with respect to t and r respectively. To continue our analysis, we consider the Lyra displacement vector to be a time-like vector as

$$\xi_\mu = (\lambda, 0, 0, 0), \tag{6}$$

where λ is a constant. Next, the field equations (1) for the spacetime metric (2) lead to the following set of equations

$$\frac{r^2 (G')^2 - m^2 G^2}{r^2 G^2} + \frac{3}{4}\lambda^2 - \frac{1}{2}(\Phi')^2 - \frac{1}{2}(\dot{\Phi})^2 = 0, \tag{7}$$

$$\frac{m^2 G^2 - r^2 (G')^2}{r^2 G^2} + \frac{3}{4}\lambda^2 + \frac{1}{2}(\Phi')^2 - \frac{1}{2}(\dot{\Phi})^2 = 0, \tag{8}$$

$$\frac{2r^2 GG'' - r^2 (G')^2 + 2rGG' - m^2 G^2}{r^2 G^2} - \frac{3}{4}\lambda^2 - \frac{1}{2}(\Phi')^2 + \frac{1}{2}(\dot{\Phi})^2 = 0. \tag{9}$$

By comparing the equations (8) and (9), one finds

$$rGG'' - r(G')^2 + GG' = 0. \tag{10}$$

The solution of this equation is of the form

$$G = ar^{n} + br^{-n}, \tag{11}$$

where *a, b* and *n* are arbitrary constants. From the equations (7) and (8), yields

$$\lambda = \pm\sqrt{\frac{2}{3}}\ \dot{\Phi}\ . \tag{12}$$

In order to solve the field equations, we assume the separable form of the massless scalar field as follows

$$\Phi(r,t) = \Phi_1(t) + \Phi_2(r). \tag{13}$$

In continuation, with the help of equations (12) and (13) together with this fact that λ is a constant, we find that

$$\Phi_1 = c_1 t + c_2, \tag{14}$$

throughout the rest of the paper, c_i are constants of integration. Therefore, we have

$$\lambda = \pm\sqrt{\frac{2}{3}}\ c_1. \tag{15}$$

By substituting equation (14) into equation (5), we conclude

$$\Phi_2 = c_3 \ln r + c_4, \tag{16}$$

On the other hand, if we substitute the equations (11-13) into equation (7), we obtain the following relation

$$\Phi_2 = \pm\sqrt{2}\int \frac{\sqrt{\left(a^2 r^{2n} + b^2 r^{-2n}\right)\left(n^2 - m^2\right) - 2ab\left(n^2 + m^2\right)}}{r\sqrt{a^2 r^{2n} + b^2 r^{-2n} + 2ab}}\, dr + c_5. \tag{17}$$

The equations (16) and (17) are equal when (1): $n = 0$ and $m = \pm\sqrt{-1}$, (2): $a(or\ b) = 0$ and $n^2 - m^2 = 1$. For the real solution, case (2) is correct. Finally, we have

$$\Phi = \pm\sqrt{2}\ln(r) + c_1 t + \Phi_0, \tag{18}$$

in which Φ_0 is a constant.

3. Classical motion of a test particle in the Bonnor metric

In this section, we will calculate the trajectory of a relativistic test particle of mass m_0 that moving in the Bonnor spacetime by using the Hamilton-Jacobi equation [6-8]. Therefore, this equation is of the form

$$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial r}\right)^2 - r^{2n^2-4}\left(\frac{\partial S}{\partial \phi}\right)^2 - r^{2n^2-4}G^4\left(\frac{\partial S}{\partial z}\right)^2 - \left(m_0 r^{n^2-1}G\right)^2 = 0. \tag{19}$$

In order to solve this first order partial differential equation, let us use separation of variables for the Hamilton-Jacobi function as follows

$$S(t,r,\phi,z) = -Et + \eta(r) + L_\phi \phi + L_z z, \tag{20}$$

here E, L_ϕ and L_z are arbitrary constants and can be identified respectively as energy and angular momentum components of test particle along ϕ and z-directions. With substituting the last relation in Hamilton-Jacobi equation, the unknown function η is given by

$$\eta = \varepsilon \int \sqrt{E^2 - r^{2n^2-4} L_\phi^2 - r^{2n^2-2} G^4 L_z^2 - r^{2n^2-2} G^2 m_0^2}\; dr, \tag{21}$$

in which $E \neq 0$ and $\varepsilon = \pm 1$ stands for the sign changing whenever r passes through a zero of the integrand, [9]. Now the equations for the trajectory can be obtained following Hamilton-Jacobi method as, [6-8]:

$$\frac{\partial S}{\partial E} = \text{constant}, \quad \frac{\partial S}{\partial L_\phi} = \text{constant}, \quad \frac{\partial S}{\partial L_z} = \text{constant}. \tag{22}$$

Consequently, after calculating and simplifying, the set of equations (22) respectively are changed to the following relations

$$t = \varepsilon E \int \frac{dr}{\sqrt{E^2 - r^{2n^2-4} L_\phi^2 - r^{2n^2-2} G^4 L_z^2 - r^{2n^2-2} G^2 m_0^2}}, \tag{23}$$

$$\phi = \varepsilon L_\phi \int \frac{r^{2m^2-2} dr}{\sqrt{E^2 - r^{2n^2-4} L_\phi^2 - r^{2n^2-2} G^4 L_z^2 - r^{2n^2-2} G^2 m_0^2}}, \tag{24}$$

$$z = \varepsilon L_z \int \frac{r^{2m^2} G^4 dr}{\sqrt{E^2 - r^{2n^2-4} L_\phi^2 - r^{2n^2-2} G^4 L_z^2 - r^{2n^2-2} G^2 m_0^2}}, \tag{25}$$

we have taken the constants in equations (22) to be zero without any loss of generality. Obviously it does not seem to be an easy task to find solutions for the above integrals. In the previous section, we found that $a(or\ b) = 0$. Hence, we will discuss the two following cases:

Case (1): $a = 0$

By defining a new variable as $u = \frac{1}{r}$ (see reference [10] for more discussion), equation (24) is transformed to

$$L_\phi^2 \left(\frac{du}{d\phi}\right)^2 u^{-2n^2+2} - E^2 u^{2n^2-2} + L_\phi^2 u^2 + b^4 L_z^2 u^{4n} + m_0^2 b^2 u^{2n} = 0. \tag{26}$$

Unfortunately, only an integral expression $\phi = \phi(u)$ can be obtained from this equation as

$$\int \frac{L_\phi du}{\sqrt{E^2 u^{4n^2-4} - L_\phi^2 u^{2n^2} - b^4 L_z^2 u^{2n^2+4n-2} - m_0^2 b^2 u^{2n^2+2n-2}}} = \pm(\phi - \phi_0), \tag{27}$$

where ϕ_0 is a constant. Calculations show that this equation can be solved exactly only for $n = 1$. So for this case, after some calculations, we can get

$$u(\phi) = \frac{\sqrt{2}E}{\zeta}\text{JacobiSN}\left(\frac{\zeta}{\sqrt{2}L_\phi}(\phi - \phi_0),\ i\delta\right), \tag{28}$$

in which $i = \sqrt{-1}$ and

$$\zeta = \sqrt{L_\phi^2 + m_0^2 b^2 + \sqrt{\left(L_\phi^2 + m_0^2 b^2\right)^2 + 4E^2 b^4 L_z^2}}, \tag{29}$$

$$\delta = \frac{-\sqrt{2}Eb^2 L_z}{\sqrt{\left(L_\phi^2 + m_0^2 b^2\right)\zeta^2 + 2E^2 b^4 L_z^2}}. \tag{30}$$

For more details about Jacobi functions, see [11] and [12]. Also, there is a constant solution obtained by setting $\frac{du}{d\phi} = 0$ in the equation (26) as

$$r_0 = \frac{\sqrt{2}b^2 L_z}{\sqrt{\zeta^2 - 2\left(L_\phi^2 + m_0^2 b^2\right)}}, \tag{31}$$

where r_0 is the radius of the circular orbit. This circular motion occurs in relativity theory just as in classical theory. In relativistic classical mechanics the finite trajectories are, in general, not closed but rather rosette shaped, [13]. On the other hand, it is easy to show that the following identity is valid

$$\text{JacobiSN}(x, 0) = \sin x, \tag{32}$$

where x is an arbitrary variable. By considering the last consequence, it is interesting to rewrite the equation (28) with $\delta = 0$. For doing this, there is only one choice, i.e. $L_z = 0$. Therefore, we shall henceforth ignore the motion in the z-direction. By applying this condition, equation (28) reduces to the following simple relation

$$u(\phi) = \frac{E}{\sqrt{L_\phi^2 + m_0^2 b^2}}\sin\left(\frac{\sqrt{L_\phi^2 + m_0^2 b^2}}{L_\phi}\left(\phi - \phi_0\right)\right). \tag{33}$$

The finite trajectories described by this equation, present a period given by

$$T = \frac{2\pi L_\phi}{\sqrt{L_\phi^2 + m_0^2 b^2}}. \tag{34}$$

From the equation (33), we can see that the trajectory of particle is bounded, i.e. the particle can be trapped by the extended object with the Bonnor geometry. Also we know that if we take zero the radial velocity, then the turning points of the trajectory are obtained. Therefore, the turning points are

$$r_{tp} = \pm \frac{\sqrt{L_{\phi}^2 + m_0^2 b^2}}{E}, \tag{35}$$

the subscript *tp* in r_{tp} refer to the turning points. Finally, with the help of equations (23) and (33), we can rewrite the trajectory of particle in terms of time as follows

$$r = \sqrt{t^2 + r_{tp}^2}, \tag{36}$$

and

$$\phi = \phi_0 + \frac{T}{2\pi} \operatorname{arccot}(\frac{t}{r_{tp}}). \tag{37}$$

Case (2) : $b = 0$

In this case, with the help of variable u, equation (24) convert to

$$L_{\phi}^2 \left(\frac{du}{d\phi} \right)^2 u^{-2n^2+2} - E^2 u^{2n^2-2} + L_{\phi}^2 u^2 + a^4 L_z^2 u^{-4n} + m_0^2 a^2 u^{-2n} = 0. \tag{38}$$

The solution of this equation is

$$\int \frac{L_{\phi} du}{\sqrt{E^2 u^{4n^2-4} - L_{\phi}^2 u^{2n^2} - a^4 L_z^2 u^{2n^2-4n-2} - m_0^2 a^2 u^{2n^2-2n-2}}} = \pm(\phi - \phi_0'), \tag{39}$$

where ϕ_0' is a constant. Similarly as in the previous case, the explicit expressions for the solutions of this equation can be obtained if only we take $n = -1$. It is clear that all the parameters of motion in this case are exactly equal to the previous case, if we exchange b with a.

4. Conclusion

In this paper, the classical behaviour of a test particle in the Bonnor spacetime based on Lyra manifold has been studied. We proved that the particle can be trapped by this gravitational field. By determining the trajectory, the period and turning points of motion have been calculated. We also showed that the parameters of motion depend on the choice of the angular momentum components of particle.

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