საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 7, №2, 2013
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 7, no. 2 , 2013

Informatics

An Optimal Control Problem for a Nonlocal Boundary Value Problem

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ABSTRACT. In the present paper the Bitsadze–Samarskii boundary value problem is considered for a quasi-linear differential equation of first order on the plane and the existence and uniqueness theorem for a generalized solution is proved; the necessary (in the linear case) and sufficient optimality conditions for optimal control problems are found. The optimal control problem is posed, where the behavior of control functions is described by elliptic-type equations with Bitsadze–Samarskii nonlocal boundary conditions. The necessary and sufficient optimality conditions are obtained in the form of Pontryagin's maximum principle and the solution existence and uniqueness theorem is proved for the conjugate problem. *© 2013 Bull. Georg. Natl. Acad. Sci.*

Key words: *Nonlocal boundary value problem, Bitsadze-Samarskii problem, generalized solution, optimal control, condition of optimality, principle of maximum.*

Nonlocal boundary value problems are quite an interesting generalization of classical problems and at the same time they are obtained in a natural manner when constructing mathematical models of real processes and phenomena occurring in physics, engineering, sociology, ecology and so on [1-3]. The Bitsadze-Samarskii nonlocal boundary value problem [4] was posed in connection with the mathematical modeling of processes of plasma physics. Its various generalizations are considered in [5-10].

In the present paper the Bitsadze-Samarskii boundary value problem is considered for a quasi-linear differential equation of first order on the plane and the existence and uniqueness theorem for a generalized solution is proved. The necessary (in the linear case) and sufficient optimality conditions for optimal control problems are found. The optimal control problem is posed, where the behavior of control functions is described by elliptic-type equations with Bitsadze–Samarskii nonlocal boundary conditions. The necessary and sufficient optimality conditions are obtained in the form of Pontryagin's maximum principle [11] and the solution existence and uniqueness theorem is proved for the conjugate problem.

1. Let G be the bounded domain on the complex plane E with the boundary Γ which is closed by a simple Liapunov curve. Denote by γ that part of the boundary Γ which is an open Liapunov curve with the parametric equation $z = z(s)$, $0 \le s \le \delta$. Let γ_0 be the diffeomorphic image of γ lying in the domain G with the parametric equation $z_0 = z_0(s)$, $0 \le s \le \delta$. Assume that γ_0 intersects Γ but not tangentially to it, $z^* \in \Gamma \setminus \gamma$ is a fixed point, $z = x + iy \in G$, $w = w_1 + iw_2$, $u = u_1 + iu_2$. Let U be some bounded subset from E. Each function $u(z) : G \to U$ will be called a control. We call the function $u(z)$ an admissible control, if $u(z) \in L_p(G)$, $p > 2$. The set of all admissible controls is denoted by Ω. Let $\partial_{\bar z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ be the generalized Sobolev derivative, $C_\alpha(\overline{G})$ be the set of all bounded functions satisfying the Helder condition with exponent α, $0 < \alpha \le 1$. The norm in $C_\alpha(\overline{G})$ is defined by the equality [12]

$$\|f\|_{C_\alpha(\overline{G})} = \max_{z\in\overline{G}} |f(z)| + \sup_{z_1, z_2 \in \overline{G}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha}.$$

It is assumed that $L_p(\overline{G})$ is the space consisting of all functions summable on $\overline{G}$ and having power $p \ge 1$. The norm in $L_p(\overline{G})$ is defined by the equality

$$\|f\|_{L_p(\overline{G})} = \left(\iint_G |f(x,y)|^p\, dxdy\right)^{\frac{1}{p}}.$$

For each fixed $u \in \Omega$ in the domain G we consider the boundary value problem [4]

$$\begin{gathered} \partial_{\bar z} w = f(z, w, \overline{w}, u),\ z \in G, \\ Re[w(z)] = \varphi(z),\ z \in \Gamma \setminus \gamma, \\ Im[w(z^*)] = c,\ z^* \in \Gamma \setminus \gamma,\ c = const, \\ Re[w(z(s))] = \sigma Re[w(z_0(s))],\ z(s) \in \gamma,\ z_0(s) \in \gamma_0,\ 0 < \sigma = const \end{gathered} \tag{1.1}$$

It is assumed that the following conditions are fulfilled.

1. The function $f(z, w, \overline{w}, u)$ is defined for $z \in G$, $|w| \le R$, $u \in U$, $f(z,0,0,0) \in L_p(G)$, $p > 2$ and performed conditions Lifshitz

$$|f(z, w, \overline{w}, u) - f(z, w_0, \overline{w}_0, u_0)| \le L(|w - w_0| + |\overline{w} - \overline{w}_0| + |u - u_0|),$$

L, R are positive constants.

2. $\varphi \in C_\alpha(\Gamma \setminus \gamma)$, $\alpha > \frac{1}{2}$.

3. There exists a number R_1, $0 < R_1 < R$, such that the inequality

$$0 < R_1[1 - 2L|G|^{1/p}(C + \|T_G\|_{C_\alpha(\overline{G})})] \le \|\varphi\|_{C_\alpha(\overline{G})}$$

is fulfilled, where $C = const > 0$, $|G| = mesG$, ϕ is the analytic function satisfying conditions (1.1), while the operator [12]

$$T_G[z, f] = -\frac{1}{\pi} \iint_G \frac{f(t)}{t - z}\, d\xi d\eta,\ t = \xi + i\eta.$$

Theorem 1. *Let conditions* 1–3 *be fulfilled. Then for each fixed* $u \in \Omega$ *the solution of problem (1.1) exists in the space* $C_\alpha(\overline{G})$ *and it is unique.*

Let us formulate the following optimal control problem [8]: Find a function $u_0(z) \in \Omega$ such that the solution of the Bitsadze–Samarskii boundary value problem (1.1) gives a minimal value to the functional

$$I(u) = \iint_G F(x, y, w_1, w_2, u_1, u_2)\, dxdy,$$

where $z = x + iy$, $w = w_1 + iw_2$, $\omega = \omega_1 + i\omega_2$.

To obtain the optimality condition, we additionally assume the following:

4. The function $f(z, w, q, u)$ is continuous with respect to w, q, u, has continuous partial derivatives $f_{w'}$ and $f_{q'}$ which are also continuous with respect to the same arguments as f. The function F is continuous with respect to w_1, w_2, u_1, u_2, continuously differentiable with respect to w_1, w_2 and belongs to the space $L_p(G)$, $p > 2$.

5. The estimates $(w, q) \in S_{wq}^{R_1} = \{(w, q) : |w|, |q| < R\}$ are valid for any $|f'_w|, |f'_q| \le N_1(R_1) < +\infty$, $|f| \le N_2(R_1) < +\infty$ in the domain G.

Theorem 2. *Let conditions* 1–5 *be fulfilled,* $u_0(z)$ *be an optimal control,* $w_0(z)$ *be the solution of problem (1.1) which corresponds to* $u_0(z)$, $\psi_0(z)$ *be the solution of the conjugate problem:*

$$\begin{gathered} \partial_{\bar z}\psi(z) + \frac{\partial f(u_0)}{\partial w}\psi(z) + \frac{\partial \bar f(u_0)}{\partial w}\bar\psi(z) = -2\partial_w F(u_0),\ \ z \in G \setminus \gamma_0, \\ Re[\psi(z)] = 0,\ \ z \in \Gamma, \\ Re[\psi(z_0^+) - \psi(z_0^-)] = \sigma Re[\psi(z)],\ \ z_0 \in \gamma_0,\ \ z \in \gamma, \end{gathered} \tag{1.2}$$

then the relation

$$\begin{aligned} &Re[f(z, w_0(z), \bar w_0(z), u_0(z))\psi_0(z) + F(z, w_0(z), \bar w_0(z), u_0(z))] = \\ &= \inf_{u \in U} Re[f(z, w_0(z), \bar w_0(z), u(z))\psi_0(z) + F(z, w_0(z), \bar w_0(z), u(z))] \end{aligned}$$

is fulfilled almost everywhere on G.

2. Let the domain G be the rectangle $\{z = x + iy : 0 \le x \le 1, 0 \le y \le 1\}$, Γ be the boundary of the domain G, $\gamma_0 = \{z_0 = x_0 + iy : 0 \le y \le 1\}$, $z^* \in \Gamma \setminus \gamma$, $c(z), d(z), B(z),\ f(z) \in L_p(\bar G),\ p > 2$, $g(z) \in C_\alpha(\Gamma \setminus \gamma)$, $0 < \alpha < 1$. For each $u \in \Omega$ in the domain G, we consider the Bitsadze-Samarskii boundary value problem

$$\begin{gathered} \partial_{\bar z} w + B(z)\bar w = f(z)u,\ z \in G, \\ Re[w(z)] = g(z),\ z \in \Gamma \setminus \gamma,\ Im[w(z^*)] = const, \\ Re[w(z)] = \sigma Re[w(z_0)],\ \ \ z \in \gamma,\ z_0 \in \gamma_0,\ 0 < \sigma = const. \end{gathered} \tag{2.1}$$

Let us consider the functional

$$I(u) = Re\iint_G [c(z)w(z) + d(z)u(z)]\, dxdy.$$

Theorem 3. *Let* $\psi_0(z)$ *be a solution of the conjugate problem*

$$\begin{gathered} \partial_{\bar z}\psi(z) - \bar B(z)\psi(z) = c(z),\ z \in G \setminus \gamma_0, \\ Re[\psi(z)] = 0,\ \ z \in \Gamma, \\ Re[\psi(z_0^+) - \psi(z_0^-)] = \sigma Re[\psi(z)],\ \ z_0 \in \gamma_0,\ z \in \gamma. \end{gathered} \tag{2.2}$$

Then for $u_0(z)$, $w_0(z)$ *to be optimal it is necessary and sufficient that the equality*

$$Re[(d(z) - \psi_0(z)f(z))u_0(z)] = \inf_{u \in U} Re[(d(z) - \psi_0(z)f(z))u(z)]$$

be fulfilled almost everywhere on G.

An algorithm and numerical solution in Mathcad for the particular case of problems (2.1) is given in [13].

ინფორმატიკა

არალოკალური სასაზღვრო ამოცანებისთვის ერთი ოპტიმალური მართვის ამოცანის შესახებ

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ნაშრომში განხილულია ოპტიმალური მართვის ამოცანები პირველი რიგის კვაზიწრფივი დიფერენციალური განტოლებებისთვის სიბრტყეზე არალოკალური სასაზღვრო პირობების შემთხვევაში. კვაზიწრფივი დიფერენციალური განტოლებისთვის დამტკიცებულია თეორემა ამონახსნის არსებობისა და ერთადერთობის შესახებ, მიღებულია ოპტიმალობის აუცილებელი პირობები მაქსიმუმის პრინციპის სახით, წრფივი ოპტიმალური მართვის ამოცანისთვის მიღებულია ოპტიმალობის აუცილებელი და საკმარისი პირობები.

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Received June, 2013