Note on Abel-Poisson Means of Trigonometric Fourier Series

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ABSTRACT. Some approximative properties of the Abel-Poisson means of trigonometric Fourier series are established. For summable functions we establish the order of deviation at certain points by above-mentioned summation means. We prove that at points, in which the indefinite integral of modulus of second order difference is estimated by the product of modulus continuity function and upper bounds of indefinite integral, it is possible to get optimal estimate. In the paper alongside with the pointwise estimate the uniform summation order is established. © 2013 Bull. Georg. Natl. Acad. Sci.

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Let us assume that \( T = [-\pi, \pi] \) and the functions \( f: \mathbb{R} \to \mathbb{R} \) are periodic with period \( 2\pi \), where \( \mathbb{R} = \mathbb{C} \). For a function \( f \in L(T) \) by \( \sigma[f] \) denote the trigonometric Fourier series of \( f \), i.e.,

\[
\sigma[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right),
\]

where

\[
a_k = a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt,
\]

\[
b_k = b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt.
\]

By \( f(x, r) \) denote Abel-Poisson means of the series \( \sigma[f] \), namely:

\[
f(x, r) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) P(r, t) \, dt, \quad 0 \leq r < 1,
\]
where
\[ P(r,t) = \frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos k t - \frac{1}{2} \frac{1-r^2}{1-2r \cos t + r^2}, \quad 0 \leq r < 1. \]  
(2)

As is known (see e.g. [1]),
\[ P(r,t) > 0, \quad t \in T. \]  
(3)
\[ P(r,t) < A \frac{1-r}{(1-r)^2 + t^2}, \quad t \in T. \]  
(4)

Denote by \( \varnothing \) the class of all functions \( \omega : [0, \pi] \to R \) with the properties:
1. \( \omega \) is continuous on \([0, \pi]\);
2. \( \omega \) is increasing;
3. \( \omega(0) = 0 \);
4. \( \omega(t) > 0, \quad 0 < t \leq \pi. \)

Below \( A(f), \ A(f, \eta), \ldots \) are positive constants depending only on the indicated parameters.

Denote \( \varphi(x,t) = f(x+t) + f(x-t) - 2f(x) \).

**Theorem 1.** Let \( f \in L(T), \ \omega \in \varnothing \) and \( 0 \leq r < 1 \). If for a point \( x \in T \)
\[ \int_0^\tau |\varphi(x,\eta)| d\eta \leq A( f, x, \tau) \omega(\tau), \quad 0 < \tau \leq \pi, \]  
(5)
then
\[ |f(x,r) - f(x)| \leq A( f, x, \eta) (1-r) \int_0^\tau \frac{\omega(t)}{t^2} dt, \quad r \geq n_0(\eta). \]  
(6)

**Proof.** Taking into account equality (2), we write
\[ \frac{1}{\pi} \int_T P(r,t) dt = 1, \quad 0 < r < 1. \]

Therefore,
\[ f(x,r) - f(x) = \frac{1}{\pi} \int_0^\pi \varphi(x,t) P(r,t) dt = \frac{1}{\pi} \int_0^\pi \varphi(x,t) P(r,t) dt + \frac{1}{\pi} \int_0^\pi \varphi(x,t) P(r,t) dt + \]  
+ \frac{1}{\pi} \int_0^\pi \varphi(x,t) P(r,t) dt = \sum_{j=1}^{\infty} M_j(f,x,r,\eta). \]  
(7)

In view of relations (2),(4),(5) and by virtue of (7), we have
\[ |M_j(f,x,r)| \leq \frac{A}{1-r} \int_0^{1-r} |\varphi(x,t)| dt \leq A( f, x) \omega(1-r). \]  
(8)
Taking into account (4), we obtain

$$ \left| M_2(f, x, r, \eta) \right| \leq A(1-r) \int_{1-r}^1 \frac{\varphi(x, t)}{t^2} dt. $$

If we apply the formula of integration by parts, then we obtain

$$ \left| M_2(f, x, r, \eta) \right| \leq A \left[ (1-r) \left[ \int_0^t \varphi(x, s) ds \right] \frac{1}{t^2} \right]_{1-r}^1 - 2(1-r) \int_{1-r}^1 \left[ \int_0^t \varphi(x, s) ds \right] \frac{dt}{t^2}. $$

Therefore, by virtue of condition (5), we can write

$$ \left| M_2(f, x, r, \eta) \right| \leq A(1-r) \int_{1-r}^1 \frac{\varphi(t)}{t^2} dt \leq A(1-r) \int_{1-r}^1 \frac{\varphi(t)}{t^2} dt. $$

(9)

If we use relation (4), then from equality (7) we conclude that

$$ \left| M_3(f, x, r, \eta) \right| \leq A(1-r) \int_{1-r}^1 \frac{\varphi(t)}{t^2} dt. $$

(10)

Taking into account relations (7), (8), (9) and (10), we obtain (6). Thus, theorem 1 is proved.

From Theorem 1 it follows

**Theorem 2.** Let $f \in L(T)$, $\varphi \in \Phi$, $[a, b] \subseteq T$, $b-a > 0$ and

$$ \sup_{a < x < b} \int_0^t \varphi(x, s) ds \leq A(f) \varphi(t), \ 0 < t \leq \pi, $$

then

$$ \sup_{a < x < b} \left| f(x, r) - f(x) \right| \leq A(f, \varphi) (1-r) \int_{1-r}^1 \frac{\varphi(t)}{t^2} dt, \quad r \geq r_0(\eta). $$

(11)

If $T = [a, b]$, then from equality (11) we obtain

$$ \| f(-, r) - f(0) \| \leq A(f) (1-r) \int_{1-r}^1 \frac{\varphi(t)}{t^2} dt, \quad r \geq r_0. $$

From this inequality it is possible to receive the corresponding result of Natanson [2].
ბუნებრივი ფენომენების ქვემოთა ახალგაზრდა შეფასები

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