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Mathematics

Some Remarks on Certain Sampling Formulas

Zurab Piranashvili

Vladimir Chavchanidze Institute of Cybernetics, Georgian Technical University, Tbilisi

(Presented by Academy Member Elizbar Nadaraya)

ABSTRACT. The sampling formulas for stochastic processes and stochastic fields are given. Certain remark about one inaccuracy is made. *© 2013 Bull. Georg. Natl. Acad. Sci.*

Key words: *stochastic process, stochastic field,* $L^{\alpha}(\Omega)$ *-process.*

If the stochastic process $\xi(t), -\infty < t < \infty$ satisfies the conditions of the theorem 6 [1], then for almost all sampling functions of the process $\xi(t)$ the formula (1) [2] is true:

$$\frac{1}{p!}\lim_{\zeta\to t}\frac{d^p}{d\zeta^p}\left(\frac{\xi(\zeta)}{(\zeta-c)^{N_0+1}\left(ae^{\delta\zeta}+be^{-\delta\zeta}\right)\sin^{N+1}(\alpha\zeta)}\left(\frac{\sin\beta(\zeta-t)}{\beta(\zeta-t)}\right)^q\right)=\sum_{k=-\infty}^{\infty}\left(\frac{(-1)^k}{\alpha}\right)^{N+1}\cdot$$

$$\cdot\left\{\sum_{\tau=0}^{N}\frac{1}{(N-\tau)!}\left[\sum_{\mu=0}^{\tau}\frac{\alpha^{\tau-\mu}}{(\tau-\mu)!}\cdot A_{\mu\tau N}\cdot\sum_{j=0}^{\mu}\frac{\xi^{(j)}\left(\frac{k\pi}{\alpha}\right)}{j!\left(t-\frac{k\pi}{\alpha}\right)}\cdot\sum_{r=0}^{\mu-j}\frac{(p+r)!(N_0+\mu-j-r)!(-1)^{p+\mu-j-r}}{r!N_0!p!(\mu-j-r)!\left(t-\frac{k\pi}{\alpha}\right)^r\left(\frac{k\pi}{\alpha}-c\right)^{N_0+\mu+1-j-r}}\right]\right.\cdot$$

$$\left.\cdot\frac{\varphi_{\tau N}(t;k,q,\alpha,\beta,\delta,a,b)}{\left(t-\frac{k\pi}{\alpha}\right)^p}\right\}+\sum_{\tau=0}^{N_0}\frac{\xi^{(\tau)}(c)}{p!(N_0-\tau)!(t-c)^{p+1}}\cdot\sum_{\mu=0}^{\tau}\frac{(p+\tau-\mu)!\psi_{\tau NN_0}(t;q,\alpha,\beta,\delta,a,b,c)}{\mu!(\tau-\mu)!(c-t)^{\tau-\mu}},\qquad(1)$$

$t\neq\frac{\nu\pi}{\alpha},\ \nu=0,\pm1,\pm2,\ldots$

for every $\alpha>\frac{\sigma}{N+1},\ 0<\beta<\frac{(N+1)\alpha-\sigma}{q},\ 0<\delta<(N+1)\alpha-\sigma-q\beta$, where N_0,N,p,q are fixed nonnegative integers, a,b,α,β,δ are positive real numbers, $c\neq 0$ is some fixed number and

$$A_{\mu\tau N}=\lim_{x\to 0}\frac{d^{\tau-\mu}}{dx^{\tau-\mu}}\left(\frac{x}{\sin x}\right)^{N+1}, \tag{2}$$

$$\varphi_{\tau N}\left(t;k,q,\alpha,\beta,\delta,a,b\right)=\lim_{\zeta\to\frac{k\pi}{\alpha}}\frac{d^{N-\tau}}{d\zeta^{N-\tau}}\left[\left(\frac{\sin\beta(\zeta-t)}{\beta(\zeta-t)}\right)^{q}\cdot\frac{1}{\left(ae^{\delta\zeta}+be^{-\delta\zeta}\right)}\right], \tag{3}$$

$$\psi_{\tau NN_0}\left(t;q,\alpha,\beta,\delta,a,b,c\right)=\lim_{\zeta\to c}\frac{d^{N_0-\tau}}{d\zeta^{N_0-\tau}}\left[\left(\frac{\sin\beta(\zeta-t)}{\beta(\zeta-t)}\right)^{q}\cdot\frac{1}{\left(ae^{\delta\zeta}+be^{-\delta\zeta}\right)\sin^{N+1}(\alpha\zeta)}\right], \tag{4}$$

When $p=N=N_0=0$, then all indexes of summation in (1) satisfies the conditions $\tau=0,\ \mu=0, j=0$ and $A_{00}=1$,

$$\varphi_{00}\left(t;k,q,\alpha,\beta,a,b,\delta\right)=\frac{1}{ae^{\delta\frac{k\pi}{\alpha}}+be^{-\delta\frac{k\pi}{\alpha}}}\cdot\left(\frac{\sin\beta\left(t-\frac{k\pi}{\alpha}\right)}{\beta\left(t-\frac{k\pi}{\alpha}\right)}\right)^{q},$$

$$\psi_{000}\left(t;k,q,\alpha,\beta,\delta,a,b,c\right)=\frac{1}{\left(ae^{\delta c}+be^{-\delta c}\right)\sin(\alpha c)}\cdot\left(\frac{\sin\beta(c-t)}{\beta(c-t)}\right)^{q}.$$

The formula (1) becomes a kind of:

$$\xi(t)=(t-c)\left(ae^{\delta t}+be^{-\delta t}\right)\sum_{k=-\infty}^{\infty}\xi\left(\frac{k\pi}{\alpha}\right)\frac{\sin\alpha\left(t-\frac{k\pi}{\alpha}\right)}{\alpha\left(t-\frac{k\pi}{\alpha}\right)}\cdot\frac{1}{\left(\frac{k\pi}{\alpha}-c\right)\left(ae^{\delta\frac{k\pi}{\alpha}}+be^{-\delta\frac{k\pi}{\alpha}}\right)}\cdot\left(\frac{\sin\beta\left(t-\frac{k\pi}{\alpha}\right)}{\beta\left(t-\frac{k\pi}{\alpha}\right)}\right)^{q}+$$
$$+\frac{\left(ae^{\delta t}+be^{-\delta t}\right)\sin(\alpha t)}{\left(ae^{\delta c}+be^{-\delta c}\right)\sin(\alpha c)}\cdot\xi(c)\left(\frac{\sin\beta(c-t)}{\beta(c-t)}\right)^{q},\qquad c\neq 0. \tag{5}$$

Obviously, that formula (5) is valid for random processes of type $L^{\alpha}(\Omega)$ under certain conditions (see [3]).

For two-dimensional stochastic field $\xi(t_1,t_2)$ formula (1) takes the form:

$$\frac{1}{p_1!\,p_2!}\lim_{\substack{\zeta_1\to t_1\\ \zeta_2\to t_2}}\frac{\partial^{p_1+p_2}}{\partial\zeta_1^{p_1}\partial\zeta_2^{p_2}}\left(\xi(\zeta_1,\zeta_2)\prod_{i=1}^{2}\frac{\left(\frac{\sin\beta_i(\zeta_i-t_i)}{\beta_i(\zeta_i-t_i)}\right)^{q_i}}{(\zeta_i-c_i)^{N_{0i}+1}\left(\left(a_ie^{\delta_i\zeta_i}+b_ie^{-\delta_i\zeta_i}\right)\sin^{N_i+1}(\alpha_i\zeta_i)\right)}\right)=$$

$$= \sum_{k_1,k_2=-\infty}^{\infty} \prod_{i=1}^{2} \left[\frac{(-1)^{k_i}}{\alpha_i} \right]^{N_i+1} \left\{ \sum_{\tau_1=0}^{N_1} \sum_{\tau_2=0}^{N_2} \frac{1}{(N_1-\tau_1)!(N_2-\tau_2)!} \sum_{\mu_1=0}^{\tau_1} \sum_{\mu_2=0}^{\tau_2} \prod_{i=1}^{2} \frac{\alpha_i^{\tau_i-\mu_i}}{(\tau_i-\mu_i)!} \cdot A_{\mu_i\tau_iN_i} \cdot \right.$$

$$\cdot \sum_{j_1=0}^{\mu_1} \sum_{j_2=0}^{\mu_2} \left[\frac{\partial^{j_1+j_2} \xi(t_1,t_2)}{\partial t_1^{j_1} \partial t_2^{j_2}} \right]_{\substack{t_1=\frac{k_1\pi}{\alpha_1} \\ t_2=\frac{k_2\pi}{\alpha_2}}} \cdot \frac{1}{j_1!j_2!\left(t_1-\frac{k_1\pi}{\alpha_1}\right)\left(t_2-\frac{k_2\pi}{\alpha_2}\right)} \cdot$$

$$\cdot \sum_{r_1=0}^{\mu_1-j_1} \sum_{r_2=0}^{\mu_2-j_2} \frac{(p_1+r_1)!(p_2+r_2)!(N_{01}+\mu_1-j_1-r_1)!(N_{02}+\mu_2-j_2-r_2)!(-1)^{p_1+p_2+\mu_1+\mu_2-j_1-r_1-j_2-r_2}}{r_1!r_2!N_{01}!N_{02}!p_1!p_2!(\mu_1-j_1-r_1)!(\mu_2-j_2-r_2)!\left(t_1-\frac{k_1\pi}{\alpha_1}\right)^{r_1}\left(t_2-\frac{k_2\pi}{\alpha_2}\right)^{r_2}} \cdot$$

$$\left. \cdot \frac{1}{\left(\frac{k_1\pi}{\alpha_1}-c_1\right)^{N_{01}+\mu_1+1-j_1-r_1}\left(\frac{k_2\pi}{\alpha_2}-c_2\right)^{N_{02}+\mu_2+1-j_2-r_2}} \right\} \cdot \frac{\varphi_{\tau_iN_i}(t_i;k_i,q_i,\alpha_i,\beta_i,a_i,b_i,\delta_i)}{\left(t_i-\frac{k_i\pi}{\alpha_i}\right)^{p_i}} +$$

$$\sum_{\tau_1=0}^{N_{01}} \sum_{\tau_2=0}^{N_{02}} \left[\frac{\partial^{\tau_1+\tau_2} \xi(t_1,t_2)}{\partial t_1^{\tau_1} \partial t_2^{\tau_2}} \right]_{\substack{t_1=c_1 \\ t_2=c_2}} \cdot \prod_{i=1}^{2} \sum_{\mu_i=0}^{\tau_i} \frac{(p_i+\tau_i-\mu_i)!\psi_{\tau_iN_iN_{0i}}(t_i;q_i,\alpha_i,\beta_i,\delta_i,a_i,b_i,c_i)}{\mu_i!(\tau_i-\mu_i)!(c_i-t_i)^{\tau_i-\mu_i}}, \tag{6}$$

$t_i \neq \frac{\nu_i\pi}{\alpha_i}$, $\nu_i = 0,\pm1,\pm2,...,$ $i=1,2$, for every $\alpha_i > \frac{\sigma_i}{N_i+1}, 0<\beta_i<\frac{(N_i+1)\alpha_i-\sigma_i}{q_i}, 0<\delta_i<(N_i+1)\alpha_i-\sigma_i-q_i\beta_i$, where $a_i,b_i,\alpha_i,\beta_i,\delta_i$, $i=1,2$ are fixed positive real numbers, c_1,c_2 are fixed real numbers. $A_{\mu\tau N}$, $\varphi_{\tau N}(t;k,q,\alpha,\beta,\delta,a,b)$, $\psi_{\tau NN_0}(t;q,\alpha,\beta,\delta,a,b,c)$ are determined by formulas (2), (3) and (4) respectively.

When $p_i=0,\ q_i=0,\ N_i=0,\ i=1,2,$ then from (6) we receive

$$\xi(t_1,t_2) = \prod_{i=1}^{2}(t_i-c_i)^{N_{0i}+1}\left(a_ie^{\delta_it_i}+b_ie^{-\delta_it_i}\right) \sum_{k_1,k_2=-\infty}^{\infty} \prod_{j=1}^{2} \frac{\sin\alpha_j\left(t_j-\frac{k_j\pi}{\alpha_j}\right)}{\alpha_j\left(t_j-\frac{k_j\pi}{\alpha_j}\right)} \cdot \frac{\xi\left(\frac{k_1\pi}{\alpha_1},\frac{k_2\pi}{\alpha_2}\right)}{\left(\frac{k_j\pi}{\alpha_j}-c_j\right)^{N_{0j}+1}\left(a_je^{\delta_j\frac{k_j\pi}{\alpha_j}}+b_je^{-\delta_j\frac{k_j\pi}{\alpha_j}}\right)} +$$

$$+\sin(\alpha_1t_1)\sin(\alpha_2t_2) \sum_{\tau_1=0}^{N_{01}} \sum_{\tau_2=0}^{N_{02}} \left[\frac{\partial^{\tau_1+\tau_2} \xi(t_1,t_2)}{\partial t_1^{\tau_1} \partial t_2^{\tau_2}} \right]_{\substack{t_1=c_1 \\ t_2=c_2}} \cdot \prod_{i=1}^{2} \sum_{\mu_i=0}^{\tau_i} \frac{\tilde{\psi}_{\tau_iN_{0i}}(\alpha_i,c_i)}{\mu_i!(c_i-t_i)^{\tau_i-\mu_i}}, \tag{7}$$

where $\psi_{\tau_iN_iN_{0i}}(\alpha_i,c_i) = \lim_{\varsigma\to c_i} \frac{d^{N_{0i}-\tau_i}}{d\varsigma^{N_{0i}-\tau_i}}\left(\frac{1}{\sin^{N_i+1}(\alpha_i\varsigma)}\right)$.

When $p_1 = p_2 = 0,\ N_1 = N_2 = 0,\ N_{01} = N_{02} = 0,$ then formula similar to (5) is obtained from (6) for two-dimensional stochastic field

$$\xi(t_1,t_2) = \prod_{i=1}^{2}(t_i - c_i)\left(a_i e^{\delta_i t_i} + b_i e^{-\delta_i t_i}\right)\cdot \sum_{k_1,k_2=-\infty}^{\infty} \xi\left(\frac{k_1\pi}{\alpha_1},\frac{k_2\pi}{\alpha_2}\right)\times$$

$$\times\prod_{j=1}^{2}\frac{1}{\left(\frac{k_j\pi}{\alpha_j}-c_j\right)\left(a_j e^{\delta_j\frac{k_j\pi}{\alpha_j}} + b_j e^{-\delta_j\frac{k_j\pi}{\alpha_j}}\right)}\cdot\frac{\sin\alpha_j\left(t_j-\frac{k_j\pi}{\alpha_j}\right)}{\alpha_j\left(t_j-\frac{k_j\pi}{\alpha_j}\right)}\cdot\left(\frac{\sin\beta_j\left(t_j-\frac{k_j\pi}{\alpha_j}\right)}{\beta_j\left(t_j-\frac{k_j\pi}{\alpha_j}\right)}\right)^{q_j} +$$

$$+\xi(c_1,c_2)\prod_{i=1}^{2}\frac{\left(a_i e^{\delta_i t_i} + b_i e^{-\delta_i t_i}\right)}{\left(a_i e^{\delta_i c_i} + b_i e^{-\delta_i c_i}\right)\sin(\alpha_i c_i)}\cdot\left(\frac{\sin\beta_i(c_i - t_i)}{\beta_i(c_i - t_i)}\right)^{q_i}. \tag{8}$$

When $p_1 = p_2 = 0,\ N_1 = N_2 = 0,\ N_{01} = N_{02} = -1$, then from (6) we receive $\psi_{\tau_i N_i N_{0i}}(t_i; q_i, \alpha_i, \beta_i, \delta_i, a_i, b_i, c_i) = 0,\ i = 1,2$ and consequently

$$\xi(t_1,t_2) = \prod_{i=1}^{2}\left(a_i e^{\delta_i t_i} + b_i e^{-\delta_i t_i}\right)\sum_{k_1,k_2=-\infty}^{\infty}\prod_{j=1}^{2}\frac{\xi\left(\frac{k_1\pi}{\alpha_1},\frac{k_2\pi}{\alpha_2}\right)}{\left(a_j e^{\delta_j\frac{k_j\pi}{\alpha_j}} + b_j e^{-\delta_j\frac{k_j\pi}{\alpha_j}}\right)}\cdot\frac{\sin\alpha_j\left(t_j-\frac{k_j\pi}{\alpha_j}\right)}{\alpha_j\left(t_j-\frac{k_j\pi}{\alpha_j}\right)}\left(\frac{\sin\beta_j\left(t_j-\frac{k_j\pi}{\alpha_j}\right)}{\beta_j\left(t_j-\frac{k_j\pi}{\alpha_j}\right)}\right)^{q_j}. \tag{9}$$

Remark. In [4] a multidimensional analogue of (9) was given incorrectly. In particular, it should be $\xi\left(\frac{k_1\pi}{\alpha_1},\frac{k_2\pi}{\alpha_2},\ldots,\frac{k_n\pi}{\alpha_n}\right)$ instead of $\xi\left(\frac{k_i\pi}{\alpha_i}\right)$ in the last formula.

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შენიშვნები ანათვლების ზოგიერთი ფორმულის შესახებ

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