**Physics**

Scattering on the Dirac Delta Potential and Reduction of the Three-Particle Problem

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ABSTRACT. The angular analysis with the aid of Jacob and Wick helicity formalism is performed in the 3-body equations of Alt-Grassberger-Sandhas-Khelashvili. In the capacity of two-particle scattering amplitudes the solutions for the Dirac delta-function like pair potentials are substituted, which are separable with respect to the initial and final linear momenta. It is shown how to reduce the problem to the system of one-dimensional integral equations. © 2013 Bull. Georg. Natl. Acad. Sci.

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It is well known that the 3-particle non-relativistic integral equations [1] are of 6-dimensional. Therefore even approximate methods of their solution are very difficult. The problem becomes comparably easier, when the pair potentials are separable in the linear momentum space. In this case the problem reduces to 3-dimensional integral equations, further reduction of which will be possible apparently by transition to the radial variables in the spherical coordinates. Then the problem becomes one-dimensional, but with infinite numbers of equations.

The problem must be most transparent in the “helicity” representation [2], where angular momenta of the two-body subsystems are considered as their total spin and the couplings to the third particle become easier.

Simultaneously Jacob and Wick [2] developed a method, which can be applied both in non-relativistic as well as relativistic cases. Moreover the masses of particles involved may be arbitrary. This is achieved by introduction of “helicity” – projection of particle’s total spin on the direction of its linear momentum.

The advantage of the helicity is provided by its invariance under the rotations of space-coordinate system. Therefore its inclusion to the total set of observables together with the system’s total angular momentum is possible. For example, one can prepare 2-particle state in their mass-centrum system by quantum numbers \((E, JM, \lambda_a, \lambda_b)\), where \(\lambda_{a,b}\) are the helicities of particles \(a\) and \(b\), correspondingly, \(E\) is system’s total energy, \(J\) - total angular momentum and \(M\) - its projection on the \(z\)-axis.
AGSK Equations in the Spherical Basis

The state vector of three particle system \( |p_x, q_x\rangle \) must be represented in the basis \( |JM; l_x, m_x; p_x, q_x\rangle \), where \( l_x, m_x \) are the angular momentum and its projection of the two-particle subsystem in its center of mass system and \( p_x, q_x \) are modulus of linear momenta. We will apply the formalism of Jacob and Wick, in which \( l_x, m_x \) plays the role of subsystem’s total momentum and helicity. Corresponding state vectors are normalized as follows

\[
\langle JM'; l'_x, m'_x; p'_x, q'_x | JM; l_x, m_x; p_x, q_x \rangle = \delta_{J', J} \delta_{M, M} \delta_{l', l} \delta_{m', m} \delta(p'_x, p_x) \delta(q'_x, q_x) \frac{1}{\sqrt{2L+1}}.
\]  

Now let us begin to the realisation of our program for AGSK [3,4] equations in the 3-particle system. The transition operators \( A_{\alpha\beta}(s) \) having a meaning of the \( S \)-matrix from channel \( \alpha \) to channel \( \beta \), satisfy the following equations:

\[
A_{\alpha\beta}(s) = -\delta_{\alpha\beta} G_0^{-1}(s) \sum_{\delta} \delta_{\alpha\delta} A_{\delta\beta}(s) G_0(s) T_\delta(s),
\]  

where \( G_0(s) = \left[H_0 - s + i\varepsilon\right]^{-1} \),

where \( H_0 \) is the total kinetic energy operator

\[
H_0 = \frac{p_x^2}{2M} + \frac{p_y^2}{2\eta_y} + \frac{q_x^2}{2\mu_x},
\]  

and here

\[
M = m_1 + m_2 + m_3; \quad \eta_y = \frac{m_2 m_3}{m_1 + m_2}; \quad \mu_x = \frac{m_1 (m_2 + m_3)}{M}.
\]  

In the 3-particle center of mass system \( P=0 \).

Two-particle subsystem operators \( T_\delta(s) \) satisfy to ordinary Lippmann-Schwinger equations

\[
T_\delta(s) = V_\delta - V_\delta G_0(s) T_\delta(s),
\]  

where \( V_\delta \) are the pair potentials between remaining two particles. This operator looks like in the above-choiced basis

\[
\langle JM'; l'_x, m'_x; p'_x, q'_x | JM; l_x, m_x; p_x, q_x \rangle = 
\delta_{J', J} \delta_{M, M} \delta_{l', l} \delta_{m', m} \delta(q'_x, q_x) \left(p'_x \right) \left(2\mu_x \right)^{-\frac{1}{2}}.
\]  

Here $t_{ps}^Q \left( s - \frac{q^2}{2\mu_s} \right)$ is a two-particle scattering amplitude satisfying to equation

$$\langle p_s | V^{ij}_s | k_s \rangle = \int k_s^2 dk_s \langle p_s | V^{ij}_s | k_s \rangle \left( \frac{1}{k_s^2 - s + i\varepsilon} \right),$$

(8)

where

$$\langle p_s | V^{ij}_s | k_s \rangle = \frac{2}{\pi} \int_0^\infty j_\ell (p_s r) \Psi'_s (r) j_\ell (k_s r) r^2 dr$$

(9)

and $j_\ell$ is the Bessel function.

Now the AGSK equations may be rewritten in the final form as follows

$$\langle l'_s m'_s ; p'_s ; q'_s | A^J_{0}\hat{M}_0 (s) | l''_s m''_s ; p''_s ; q''_s \rangle =$$

$$-\sum_s \sum_{l_I m_I} \hat{\delta}_{l_I} \hat{\delta}_{m_I} \int_0^\infty \frac{p^2_s}{2\eta_s} + \frac{q^2_s}{2\mu_s} \left( s - \frac{q^2_s}{2\mu_s} \right) \langle l'_s m'_s ; p'_s ; q'_s | l''_s m''_s ; p''_s ; q''_s \rangle \left( \frac{p^2_s}{2\eta_s} \right) \left( s - \frac{q^2_s}{2\mu_s} \right),$$

(10)

where the “recoupling coefficients” are substituted by the following relation

$$\langle J^M ; l'_s m'_s ; p'_s ; q'_s | A^J_{0}\hat{M}_0 (s) | l''_s m''_s ; p''_s ; q''_s \rangle = \delta_{l'_s} \delta_{m'_s} \delta_{l''_s} \delta_{m''_s} \langle l'_s m'_s ; p'_s ; q'_s | l''_s m''_s ; p''_s ; q''_s \rangle.$$

(11)

The equations (10) are valid for any pair potentials between two particles.

We see that in radial representation, when integrations are now by moduli of linear momenta, derived integral equations are two-dimensional. For further reduction of these equations we need to specify the pair potentials, which will be done in the following section.

**Delta-Function-Like Pair Potentials**

As an example let us consider the following two-body potentials:

$$V_\delta (r) = V_\delta^0 \delta (r - a_\delta).$$

(12)

These potentials represent impenetrable spheres — “bubbles”, interaction to which takes place on some point of its surface. Because interaction occurs only at one point, they belong to the class of local potentials by the direct meaning. Such a potential was considered by E.Fermi [5] in nuclear physics. Certainly such potentials do not have a deep physical meaning. For us they are interesting because, as will be seen below, two-body amplitudes become separable ones according to linear momenta. Therefore they represent a rare exception.
among all local potentials. Using (12) in Eq.(9), it follows

$$\langle p_\delta | V_{ij}^\delta | p_\delta \rangle = \frac{2V_{ij}^\delta a_\delta^2}{\pi} \int \left( p a_\delta \right) j_{i_j}^\delta \left( p a_\delta^j \right).$$

Then after its substitution into the Lippmann-Schwinger equation (8), the last equation is easily solved [5]

$$\langle p_\delta | t_{ij}^\delta (s) | p_\delta \rangle = \frac{2V_{ij}^\delta a_\delta^2}{\pi} \int \left( p a_\delta \right) j_{i_j}^\delta \left( p a_\delta^j \right) t_{i_j}^\delta (s) = \langle p_\delta | V_{ij}^\delta (s) | p_\delta \rangle t_{i_j}^\delta (s),$$

where

$$t_{i_j}^\delta (s) = \left\{ \begin{array}{l} 1 + \frac{2V_{ij}^\delta a_\delta^2}{\pi} \int_0^{\infty} \frac{j_{i_j}^\delta (ka_\delta)}{k^2 - s + i\epsilon} \end{array} \right\}^{-1}.$$ \( \text{(15)} \)

Now the equation (10) will be simplified also and it reduces to the 1-dimensional system of equations

$$A_{ij}^{M \delta} \left( s, l_j m_j, q_\delta \right) = -\sum_{\delta_1} \int_0^{\infty} p_{\delta_1}^2 dp_{\delta_1} j_{\delta_1} \left( p_{\delta_1} a_\delta \right) \langle \hat{\theta} : m_{\delta_1} p_{\delta_1}^j q_{\delta_1}^j : \rangle \langle \hat{l}_j : m_j p_j q_j : \rangle,$$

$$-\sum_{\delta_2} \sum_{l_m} \tilde{\delta}_{\delta_1 \delta_2} \int_0^{\infty} A_{ij}^{M \delta_2} \left( s, l_j m_j, q_\delta \right) B \left( q_\delta^j l_j : p_{\delta_2}^j \right) q_{\delta_2}^2 dq_{\delta_2}^j,$$ \( \text{(16)} \)

where it is denoted

$$A_{ij}^{M \delta} \left( s, l_j m_j, q_\delta \right) = \left\{ \begin{array}{l} 0 \langle \hat{l}_j : m_{\delta_1} p_{\delta_1}^j q_{\delta_1}^j : \rangle \langle \hat{\theta} : m_{\delta_1} p_{\delta_1}^j q_{\delta_1}^j : \rangle \frac{j_{\delta_1} \left( p_{\delta_1} a_\delta \right) p_{\delta_1}^2 dp_{\delta_1}}{q_{\delta_1}^2 + p_{\delta_1}^2} + \frac{p_{\delta_1}^2}{2\mu_{\delta_1}^2} - s + i\epsilon \end{array} \right\}.$$ \( \text{(17)} \)

and the effective potential \( B \) is to be obtained from the equation

$$B \left( q_\delta^j l_j : m_j \right) = \frac{2V_{ij}^\delta a_\delta^2}{\pi} \int_0^{\infty} \frac{p_{\delta_1}^2 dp_{\delta_1} j_{\delta_1} \left( p_{\delta_1} a_\delta \right) p_{\delta_1}^2 dp_{\delta_1} \langle \hat{\theta} : m_{\delta_1} p_{\delta_1}^j q_{\delta_1}^j : \rangle \langle \hat{l}_j : m_j p_j q_j : \rangle \rangle j_{\delta_1} \left( p_{\delta_1} a_\delta \right) \frac{p_{\delta_1}^2}{2\mu_{\delta_1}^2} + \frac{q_{\delta_1}^2}{2\mu_{\delta_1}^2} - s + i\epsilon.$$ \( \text{(18)} \)

At the end we have derived the system of infinite numbers of 1-dimensional integral equations (16). If we take into account that only finite numbers of \( l_j \) momenta will contribute to the two-body amplitudes, then our problem reduces to the comprehended one and the obtain equations may be studied with sufficient accuracy.

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