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Physics

Scattering on the Dirac Delta Potential and Reduction of the Three-Particle Problem

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ABSTRACT. The angular analysis with the aid of Jacob and Wick helicity formalism is performed in the 3-body equations of Alt-Grassberger-Sandhas-Khelashvili. In the capacity of two-particle scattering amplitudes the solutions for the Dirac delta-function like pair potentials are substituted, which are separable with respect to the initial and final linear momenta. It is shown how to reduce the problem to the system of one-dimensional integral equations. *© 2013 Bull. Georg. Natl. Acad. Sci.*

Key words: *delta potential, Jacob and Wick helicity, three body problem.*

It is well known that the 3-particle non-relativistic integral equations [1] are of 6-dimensional. Therefore even approximate methods of their solution are very difficult. The problem becomes comparably easier, when the pair potentials are separable in the linear momentum space. In this case the problem reduces to 3-dimensional integral equations, further reduction of which will be possible apparently by transition to the radial variables in the spherical coordinates. Then the problem becomes one-dimensional, but with infinite numbers of equations.

The problem must be most transparent in the "helicity" representation [2], where angular momenta of the two-body subsystems are considered as their total spin and the couplings to the third particle become easier.

Simultaneously Jacob and Wick [2] developed a method, which can be applied both in non-relativistic as well as relativistic cases. Moreover the masses of particles involved may be arbitrary. This is achieved by introduction of "helicity" – projection of partile's total spin on the direction of its linear momentum.

The advantage of the helicity is provided by its invariance under the rotations of space-coordinate system. Therefore its inclusion to the total set of observables together with the system's total angular momentum is possible. For example, one can prepare 2-particle state in their mass-centrum system by quantum numbers $(E, JM, \lambda_a, \lambda_b)$, where $\lambda_{a,b}$ are the helicities of particles *a* and *b*, correspondingly, *E* is system's total energy, *J* - total angular momentum and *M* - its projection on the *z*-axis.

AGSK Equations in the Spherical Basis

The state vector of three particle system $|\boldsymbol{p}_\alpha, \boldsymbol{q}_\alpha\rangle$ must be represented in the basis $|JM; l_\alpha m_\alpha; p_\alpha; q_\alpha\rangle$, where l_α, m_α are the angular momentum and its projection of the two-particle subsystem in its center of mass system and p_α and q_α are modulus of linear momenta. We will apply the formalism of Jacob and Wick, in which l_α, m_α plays the role of subsystem's total momentum and helicity. Corresponding state vectors are normalized as follows

$$\langle J'M'; l'_\alpha m'_\alpha; p'_\alpha; q'_\alpha | JM; l_\alpha m_\alpha; p_\alpha; q_\alpha\rangle = \delta_{JJ'}\delta_{MM'}\delta_{l_\alpha l'_\alpha}\delta_{m_\alpha m'_\alpha}\delta(p_\alpha - p'_\alpha)p_\alpha^{-2}\delta(q_\alpha - q'_\alpha)q_\alpha^{-2}. \quad (1)$$

Now let us begin to the realisation of our program for AGSK [3,4] equations in the 3-particle system. The transition operators $A_{\alpha\beta}(s)$ having a meaning of the S - matrix from channel α to chennel β ,satisfy the following equations:

$$A_{\alpha\beta}(s) = -\bar{\delta}_{\alpha\beta}G_0^{-1}(s) - \sum_\delta \bar{\delta}_{\alpha\delta}A_{\beta\delta}(s)G_0(s)T_\delta(s), \quad (2)$$

here $\bar{\delta}_{\alpha\beta} \equiv (1-\delta_{\alpha\beta})$ and indices $\alpha, \beta, \delta = 0,1,2,3$ characterize the asymptotic states in three particle system (e.g. $\alpha = 0$ means, that all 3 particles are free, $\alpha = 1$ means that the 2nd and 3rd particles compose a bound state and 1st particle is free, and so on). $G_0(s)$ is the free Green's function

$$G_0(s) = [H_0 - s + i\varepsilon]^{-1}, \quad (3)$$

where H_0 is the total kinetic energy operator

$$H_0 = \frac{\mathrm{P}^2}{2M} + \frac{p_\alpha^2}{2\eta_\alpha} + \frac{q_\alpha^2}{2\mu_\alpha} \quad (4)$$

and here

$$M = m_1 + m_2 + m_3; \qquad \eta_\alpha = \frac{m_\beta m_\gamma}{m_\beta + m_\gamma}; \qquad \mu_\alpha = \frac{m_\alpha(m_\beta + m_\gamma)}{M}. \quad (5)$$

In the 3-particle center of mass system P=0.

Two-particle subsystem operators $T_\delta(s)$ satisfy to ordinary Lippmann-Schwinger equations

$$T_\delta(s) = V_\delta - V_\delta G_0(s)T_\delta(s), \quad (6)$$

where V_δ are the pair potentials between remaining two particles. This operator looks like in the above-choiced basis

$$\langle J'M'; l'_\delta m'_\delta p'_\delta; q'_\delta | T_\delta(s) | JM; l_\delta m_\delta p_\delta; q_\delta\rangle =$$
$$= \delta_{JJ'}\delta_{MM'}\delta_{l_\delta l'_\delta}\delta_{m_\delta m'_\delta}q_\delta^{-2}\delta(q_\delta - q'_\delta)\langle p'_\delta | t_\delta^{l_\delta}\left(s - \frac{q_\delta^2}{2\mu_\delta}\right) | p_\delta\rangle. \quad (7)$$

Here $t_\delta^{l_\delta}\left(s-\dfrac{q_\delta^2}{2\mu_\delta}\right)$ is a two-particle scattering amplitude satisfying to equation

$$\left\langle p_\delta \left| t_\delta^{l_\delta}(s) \right| p'_\delta \right\rangle = \left\langle p_\delta \left| \left\langle p_\delta \left| V_\delta^{l_\delta} \right| p'_\delta \right\rangle - \int_0^\infty \frac{k_\delta^2 dk_\delta \left\langle p_\delta \left| \left\langle p_\delta \left| V_\delta^{l_\delta} \right| k_\delta \right\rangle \left\langle k_\delta \left| t_\delta^{l_\delta}(s) \right| p'_\delta \right\rangle}{\dfrac{k_\delta^2}{2\eta_\delta} - s + i\varepsilon}, \quad (8)$$

where

$$\left\langle p_\delta \left| V_\delta^{l_\delta} \right| k_\delta \right\rangle = \frac{2}{\pi}\int_0^\infty j_{l_\delta}(p_\delta r) V_\delta(r) j_{l_\delta}(k_\delta r) r^2 dr \quad (9)$$

and j_l is the Bessel function.

Now the AGSK equations may be rewriten in the final form as follows

$$\begin{aligned} &\left\langle l'_\alpha m'_\alpha; p'_\alpha; q'_\alpha \left| A_{\alpha\beta}^{JM}(s) \right| l_\beta m_\beta; p_\beta; q_\beta \right\rangle = \\ &= -\bar{\delta}_{\beta\alpha}\left(\frac{p_\beta^2}{2\eta_\beta} + \frac{q_\beta^2}{2\mu_\beta} - s\right)\left\langle l'_\alpha m'_\alpha; p'_\alpha; q'_\alpha \left| l_\beta m_\beta; p_\beta; q_\beta \right. \right\rangle - \\ &-\sum_\delta \sum_{l''_\delta m''_\delta} \bar{\delta}_{\beta\delta} \int_0^\infty \left\langle l'_\alpha m'_\alpha; p'_\alpha; q'_\alpha \left| A_{\alpha\delta}^{JM}(s) \right| l''_\delta m''_\delta; p''_\delta; q''_\delta \right\rangle p''^2_\delta dp''_\delta q''^2_\delta dq''_\delta \cdot \\ &\cdot \left(\frac{q''^2_\delta}{2\mu_\delta} + \frac{p''^2_\delta}{2\eta_\delta} - s + i\varepsilon\right)^{-1} \left\langle p''_\delta \left| t_\delta^{l''_\delta}\left(s - \frac{q''^2_\delta}{2\mu_\delta}\right) \right| p'''_\delta \right\rangle p'''^2_\delta dp'''_\delta \times \\ &\times \left\langle l''_\delta m''_\delta; p''_\delta; q''_\delta \left| l_\beta m_\beta; p_\beta; q_\beta \right. \right\rangle, \end{aligned} \quad (10)$$

where the “recoupling coefficients” are substituted by the following relation

$$\left\langle J'M'; l'_\alpha m'_\alpha; p'_\alpha; q'_\alpha \left| JM; l_\beta m_\beta; p_\beta; q_\beta \right. \right\rangle = \delta_{JJ'}\delta_{MM'}\left\langle l'_\alpha m'_\alpha; p'_\alpha; q'_\alpha \left| l_\beta m_\beta; p_\beta; q_\beta \right. \right\rangle. \quad (11)$$

The equations (10) are valid for any pair potentials between two particles.

We see that in radial representation, when integrations are now by moduli of linear momenta, derived integral equations are two-dimensional. For further reduction of these equations we need to specify the pair potentials, which will be done in the following section.

Delta-Function-Like Pair Potentials

As an example let us consider the following two-body potentials:

$$V_\delta(r) = V_\delta^0 \delta(r - a_\delta). \quad (12)$$

These potentials represent inpenetrable spheres –”bubbles”, interaction to which takes place on some point of its surface. Because interaction occurs only at one point, they belong to the class of local potentials by the direct meaning. Such a potential was consider still by E.Fermi [5] in nuclear physics. Certenly such potentials do not have a deep physical meaning. For us they are interesting because, as will be seen below, two-body amplitudes become separable ones according to linear momenta. Therefore they represent a rare excepton among all local potentials. Using (12) in Eq.(9), it follows

$$\langle p_\delta | V_\delta^{l_\delta} | p'_\delta \rangle = \frac{2V_\delta^0 a_\delta^2}{\pi} j_{l_\delta}(pa_\delta) j_{l_\delta}(p'a_\delta). \tag{13}$$

Then after its substitution into the Lippmann-Schwinger equation (8), the last equation is easily solved [5]

$$\langle p_\delta | t_\delta^{l_\delta}(s) | p'_\delta \rangle = \frac{2V_\delta^0 a_\delta^2}{\pi} j_{l_\delta}(pa_\delta) j_{l_\delta}(p'a_\delta) \tau_{l_\delta}(s) = \langle p_\delta | V_\delta^{l_\delta}(s) | p'_\delta \rangle \tau_{l_\delta}(s), \tag{14}$$

where

$$\tau_{l_\delta}(s) = \left\{ 1 + \frac{2V_\delta^0 a_\delta^2}{\pi} \int_0^\infty \frac{j_{l_\delta}^2(ka_\delta)}{\frac{k^2}{2\eta_\delta} - s + i\varepsilon} \right\}^{-1}. \tag{15}$$

Now the equation (10) will be simplified also and it reduces to the 1-dimensional system of equations

$$A_{\alpha\beta}^{JM}(s, l_\beta m_\beta; q_\beta) = -\bar{\delta}_{\beta\alpha} \int_0^\infty p_\beta^2 dp_\beta j_{l_\beta}(p_\beta a_\beta) \langle l'_\alpha m'_\alpha p'_\alpha; q'_\alpha | l_\beta m_\beta p_\beta; q_\beta \rangle -$$

$$- \sum_\delta \sum_{l''_\delta m''_\delta} \bar{\delta}_{\beta\delta} \int_0^\infty A_{\alpha\delta}^{JM}(s, l''_\delta m''_\delta; q''_\delta) B(q''_\delta l''_\delta; p''_\delta) q''^2_\delta dq''_\delta, \tag{16}$$

where it is denoted

$$A_{\alpha\beta}^{JM}(s, l_\beta m_\beta; q_\beta) \equiv \int_0^\infty \langle l_\alpha m_\alpha p_\alpha; q_\alpha | l''_\beta m''_\beta p''_\beta; q''_\beta \rangle \cdot \frac{j_{l''_\beta}(p''_\beta a_\beta) p''^2_\beta dp''_\beta}{\frac{q''^2_\beta}{2\mu_\beta} + \frac{p''^2_\beta}{2\eta_\beta} - s + i\varepsilon} \tag{17}$$

and the effective potential B is to be obtained from the equation

$$B(q''_\delta l''_\delta m''_\delta) = \frac{2V_\delta^0 a_\delta^2}{\pi} \tau_{l''_\delta}(q''_\delta) \cdot \int_0^\infty \frac{p_\beta^2 dp_\beta j_{l''_\delta}(p'''_\delta a_\delta) p'''^2_\delta dp'''_\delta \langle l''_\delta m''_\delta; p''_\delta; q''_\delta | l_\beta m_\beta; p_\beta; q_\beta \rangle j_{l_\beta}(p_\beta a_\beta)}{\frac{p_\beta^2}{2\eta_\beta} + \frac{q_\beta^2}{2\mu_\beta} - s + i\varepsilon}. \tag{18}$$

At the end we have derived the system of infinite numbers of 1-dimensional integral equations (16). If we take into account that only finite numbers of l_δ momenta will contribute to the two-body amplitudes, then our problem reduces to the comprehended one and the obtain equations may be studied with sufficient accuracy.

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