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Mathematics

Remarks on Bicentric Polygons

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ABSTRACT. We consider one-dimensional families of bicentric polygons with the fixed incircle and circumcircle. The main attention is paid to the topology of moduli spaces of associated linkages and to the extremal values of area of bicentric *n*-gons. For even *n* we establish that moduli spaces of bicentric *n*-gons have singular points of quadratic type and give an exact upper estimate for the number of singular points. We also indicate certain restrictions on the possible values of Euler characteristics of moduli spaces and discuss its possible changes in families of bicentric polygons. For *n*=6, 8 we give an estimate for the number of critical points of area in a family of bicentric *n*-gons and describe the shape of extremal polygons. Moreover, we calculate the mean value of area for *n*=3. A number of the other results in a few concrete cases are also established and two plausible conjectures are formulated. *© 2013 Bull. Georg. Natl. Acad. Sci.*

Key words: *bicentric polygon, Euler triangle formula, Fuss relation, generalized Fuss relations, polygonal linkage, configuration space, shape space.*

1. A polygon is called *bicentric* if it has an inscribed circle (incircle) and circumscribed circle (circumcircle) [1]. Recall that a one-dimensional family of bicentric polygons can be associated with certain pairs of circles (C,S) such that C lies inside S. To this end it is sufficient to assume that there exists a bicentric n-gon associated with the (C,S) pair, i.e., an n-gon inscribed in S and circumscribed about C. Then, by the famous Poncelet closure theorem (Poncelet porism), there exists a one-dimensional family $B(C,S)$ of bicentric n-gons accociated with (C, S) [1]. These n-gons are called *bicentric polygons associated with* (C,S) and pair (C,S) is called the *frame* of this family. In such case we say that (C,S) is a *Poncelet pair of circles of order* n.

Up to a rigid motion such a pair of circles is determined by a triple of non-negative numbers (R,r,d) satisfying $R>r>d$, $R>d\geq 0$, where R is the radius of external circle S; r is the radius of internal circle C, and d is the distance between the centers of these circles. The triple (R,r,d) is called the *gauge triple* of (C,S). It is known that in order that pair (C,S) is a Poncelet pair of order n, the numbers R,r,d should satisfy a certain algebraic relation $F_n(R,r,d)=0$ [1].

For example, for $n=3$, this is the *Euler triangle formula* $R^2-2Rr=d^2$, and for $n=4$, this is the so-called *Fuss relation:*

$$(R-d)^{-2}+(R+d)^{-2}=r^{-2}.$$

For small values of n, it is known that these relations, called *generalized Fuss relations (GFR) of order* n, are in fact criteria, i.e., they guarantee that a given pair of circles is a Poncelet pair of order n. To present the next two GFR in concise form it is convenient to put $p=R+d$, $q=R-d$. Then the generalized Fuss relation of order five can be written as

$$p^3q^3+p^2q^2r(p+q)-pqr^2(p+q)^2-r^3(p+q)(p-q)^2=0.$$

For $n=6$ it reads

$$3p^4q^4-2p^2q^2r^2(p^2+q^2)-r^4\left(p^2-q^2\right)^2=0.$$

We will also consider $n=8$, where GFR reads

$$\frac{1}{\left(p^2q^2-4Rr^2d\right)^2}+\frac{1}{\left(p^2q^2+4Rr^2d\right)^2}=\frac{1}{\left(2r^2(R^2+d^2)-p^2q^2\right)^2}.$$

The last two relations can be obtained from the Euler triangle formula and Fuss relation using a recent general result given in [2].

Proposition 1. For $n>2$, $F_{2n}(R,r,d)=F_n(R,r_1,d_1)$, where

$$r_1=\frac{2Rr^2\left(R^2+d^2\right)}{\left(R^2-d^2\right)^2},\quad d_1=\frac{4R^2r^2d}{\left(R^2-d^2\right)^2}.$$

This result will also be used to derive further consequences in the sequel.

Using scaling one can without loss of generality assume that $R=1$. Then from the above relations it follows that for each value of d one can use them to find the values of r corresponding to pairs of Poncelet circles of order n. In other words, for each n the set of Poncelet pairs of circles of order n is one-dimensional.

Given a Poncelet pair of circles (C,S) there arise a number of natural problems concerned with expressing various numerical invariants of the associated bicentric polygons in terms of its gauge triple (R,r,d). Problems of such type have been considered, e.g., in [3-5]. In this note, we investigate the arising families of moduli spaces and critical points of area in families of bicentric polygons. To this end we use a recent general result (Proposition 1) established in [2].

The results on critical points of area (Theorems 3, 4) are given for $n=6,8$. Analogs of these results for odd $n\geq 5$ will be discussed in a forthcoming detailed paper. We also give a new result for $n=3$. Notice that each triangle is bicentric and yields a family of triangles associated with its incircle and circumcircle. We calculate the mean value of area for this family of triangles (Proposition 2).

In the next section we place some of the arising problems in the context of polygonal linkages. For this reason we begin with brief recollections about polygonal linkages and their moduli spaces.

2. We parameterize the family of bicentric polygons by the polar angle t of the first vertex $v_1\in S$ and assume that $n=2k$ is even. Thus we consider a family of bicentric $2k$-gons $\{P_t,\ t\in[0,2\pi]\}$ with the frame

(C,S) having the gauge triple (R,r,d). If $d=0$, the circles are concentric and all bicentric polygons are congruent regular 2*k*-gons.

Our first goal is to investigate the planar configuration spaces $C(L_t)$ of polygonal linkages $L_t = L(P_t)$ associated with bicentric polygons. Recall that a polygonal linkage L is defined by a *n*-tuple of positive numbers l_i called *sidelengths* of L [6]. In case of a closed polygonal linkage it is always assumed that each of the sidelengths is not greater than the sum of all other ones. Notice that each *n*-gon P canonically defines a *n*-gonal linkage $L(p)$ with the sidelengths equal to the lengths of the sides of P. A polygonal linkage is called *regular* if all the sidelengths are equal.

The *planar configuration space* $C(L)$ of a n-gonal linkage L is defined as the collection of all *n*-tuples of points v_i in Euclidean plane such that the distance between v_i and v_{i+1} is equal to l_i, where it is assumed that $v_{n+1} = v_1$. Factoring $C(L)$ over the natural diagonal action of the group $E(2)$ of orientation preserving motions of the plane, one obtains the *moduli space* $M(L)$ often called also the *shape space* of L. Shape spaces, as well as configuration spaces, are endowed with the natural topologies induced by Euclidean metric.

It is easy to see that the planar shape space can be identified with the subset of configurations such that $v_1 = (0,0),\ v_2 = (l_1,0)$. It is well known that for a closed *n*-chain the moduli space has a natural structure of compact orientable real-algebraic set of dimension $k-3$. Let us say that a polygonal linkage is *degenerate* if it has an *aligned configuration*, i.e., a configuration where all vertices lie on the same straight line. It is well known that this happens if and only if there exists a *k*-tuple of "signs" $s_i = \pm 1$ such that $\sum s_i l_i = 0$. The shape space $M(L)$ of polygonal linkage L is smooth (does not have singular points) if and only if L is nondegenerate (see, e.g., [6]).

We now fix (C,S) and denote by $M(t)$ the shape space of polygonal linkage $L(t)$ defined by polygon P_t.

Theorem 1. *For $n=2k$, each shape space $M(t)$ is a compact algebraic variety with nonempty singular set consisting of isolated singular points of quadratic cone type. The number m of singular points of $M(t)$ satisfies inequalities $1 \le m \le K_{2k}$, where K_{2k} is the number of different aligned configurations of a regular 2k-linkage, and both these estimates are realized on the set of bicentric 2k-gons.*

The proof follows from several observations. First, it is easy to see that the lengths of the sides of tangential 2*k*-gon satisfy relation "the sum of odd-numbered ones is equal to the sum of even-numbered ones". According to [6] this implies that the shape space is singular. It was also shown in [6] that its singular points are of quadratic cone type. As was shown in [6], the number of singular points of shape space is maximal for regular 2*k*-linkage and it is equal to the number K_{2k} of different aligned configurations of a regular 2*k*-linkage. Since for $d=0$ the associated linkages are regular we conclude that the upper estimate is exact. For d very close to R, polygon P_0 has two "long sides" in the sense of [6]. In [6] it is shown that in such case the number of singular points equals one, which implies that both estimates are exact and completes the proof.

Recursive formulae for K_n are given in [7]. For $n = 4,6$ one can obtain a complete description of possible values of *m* by considering concrete examples. For $n=4$, it turns out that the number of singular points of $M(t)$ can be 1, 2 or $3 = K_4$. For $n=6$, the number of singular points can be any number in the segment $[1, K_6 = 10]$. These facts and some similar examples for arbitrary even *n* suggest the following conjecture.

Conjecture 1. For any even $n \geq 4$, the number of singular points of $M(t)$ can attain any natural value in the segment $[1, K_n]$.

Notice that for odd $n \geq 5$ the shape space of bicentric polygon can be smooth. Indeed, for $n = 5$ and $d = 0$ one gets the shape space of regular pentagon which is a smooth compact orientable two-dimensional surface of genus 4 [6]. Moreover, for $n = 5$ and $d > 0$ the topological type of moduli space is non constant on the set of bicentric pentagons, which implies that the moduli spaces can be smooth for certain *t* and singular for other values of *t*. Thus the description of shape spaces for odd *n* is essentially more difficult and requires a separate discussion.

Another natural invariant arising in our setting is the Euler characteristic $\chi(t) = \chi(L(t))$ of the shape space of $L(t)$. The following analog of Theorem 1 for the possible values of Euler characteristics can be proven using results of [7].

Theorem 2. *For n=2k the absolute value of the Euler characteristics $\chi(L(t))$ does not exceed C_k^{2k-1}.*

For small *n* these results can again be explicated by considering examples. In this way it can be shown that the Euler characteristic of bicentric quadrilateral can only take values $-1, -2, -3$. For $n = 6$, the values of Euler characteristic can be any integer from the segment [1,10]. It is interesting to verify the analog of Conjecture 1, namely, that the possible values of Euler characteristic fill a whole integer interval. Results of [7] can be used to clarify this issue but we do not discuss it here and proceed by considering the critical points of area in families of bicentric *n*-gons.

3. We deal now with the area $A(P_t)$ considered as a function of $t \in [0, 2\pi]$ and aim at counting and identifying its critical points. To this end it appears helpful to use different parameterization of bicentric polygons suggested in [3]. Let us denote by the length of tangent to C from the first vertex of bicentric polygon. It is easy to see that s can be used as another parameter of bicentric polygons. As was shown in [3], the possible values of s belong to the segment $[s_m, s_M]$, where

$$s_m = \sqrt{(R-d)^2 - r^2}, \quad s_M = \sqrt{(R+d)^2 - r^2}.$$

We put $A(s) = A(P_s)$ and aim at estimating the number of critical points and determining the absolute minimum and maximum of the function $A(s)$.

For small values of *n*, these problems have been considered in a number of papers, e.g., in [3-5], to name just a few. In particular, the values of absolute extrema and the shape of extremal bicentric n-gons have been found for $n \leq 4$ [3, 4]. For example, for $n = 3$ it was shown that the absolute minimum of area is attained for $s = s_m$ while the absolute maximum is attained for $s = s_M$. For $n = 4$, one has similar results [4]. Given these facts the extremal values of $A(s)$ can be found using the high school geometry. Moreover, it is shown that, for $n = 3$, the number of critical points of $A(s)$ in $[s_m, s_M]$, is four, while for $n = 4$ it is equal to five.

The proofs in [3, 4] are based on finding explicit formulae for $A(s)$ and finding critical points from the equation $A'(s) = 0$. This approach meets serious difficulties for bigger values of n, especially for odd n, and failed already for $n = 4$. We managed to partially overcome these difficulties for $n = 6$ and $n = 8$ using symmetry arguments and results of [2]. Namely, the symmetry arguments suggest several values of s as candidates for the critical points of $A(s)$. Giving an infinitesimal increment to the argument at conjectural critical point and estimating the first order terms in the expansion of area it appears possible to verify that they are indeed critical and establish the type of extremum. The results obtained in this way are less complete than for $n = 3, 4$ but seem to be instructive and provide certain evidence to a general conjecture formulated below.

Theorem 3. *For $n = 6$, the area $A(P_s)$ has non-degenerate local minimum at $s = s_m$ and non-degenerate local maximum at $s = s_M$. The number of critical points of $A(s)$ in $[s_m, s_M]$ is not less than 7.*

Theorem 4. *For $n = 8$, the area $A(P_s)$ has non-degenerate local minimum at $s = s_m$ and non-degenerate local maximum at $s = s_M$. The number of critical points of $A(s)$ in $[s_m, s_M]$ is not less than 9.*

We are unable to prove neither that $A(s)$ has no other critical points nor that the values of $A(s)$ at $s = s_m$ and $s = s_M$ are the global minimum and maximum respectively. It appeared also difficult to obtain explicit expression for the conjectural values of global minimum and maximum. However, we have some good evidence that these facts are true even in bigger generality and we wish to formulate a general conjecture suggested by our observations.

Conjecture 2. For n=2k, the absolute minimum of area is attained for $s = s_m$, while the absolute maximum is attained for $s = s_M$. The number of critical points of $A(P_s)$ equals n+1. The values of absolute minimum and maximum can be calculated as the minimal and maximal positive roots of a univariate polynomial with coefficients algebraically expressible through R, r, d [8].

It is very likely that the last statement can be proven analogously to a similar statement on critical values given in [8].

For $n = 3$, one can also find the mean value of area of bicentric polygons. Let us fix a gauge triple (R, r, d) of order three. According to [3], the domain of parameter s is $[s_m, s_M]$ given above and, for s in this interval, the area of triangle P_s is

$$r\left(s + \frac{4Rrs}{r^2 + s^2}\right).$$

Thus, to find the mean value MV of area it is sufficient to take the integral of this expression and divide it by the length of the above interval. In this case the integral can be computed explicitely and we obtain:

Proposition 2. $$MV = \frac{2Rr\left(d + 2r\ln\frac{R+d}{R-d}\right)}{\sqrt{(R+d)^2 - r^2} - \sqrt{(R-d)^2 - r^2}}.$$

Notice that in the limit case where $d = 0$ we get the correct answer $3r^2\sqrt{3} = \frac{3}{4}R^2\sqrt{3}$ since in this case all P_s are regular triangles with the side of length $2r\sqrt{3} = R\sqrt{3}$. It is also easy to show that when d tends to R the limit of this expression is zero, which again fits the intuitive expectation.

For $n \geq 4$, the arising integrals cannot be expressed in quadratures but one can still find the limits of the mean value of area in two extremal cases using expansion in power series.

In conclusion we add that analogs of the above results held for bicentric polygons with odd number of sides and Poncelet polygons associated with pairs of confocal ellipses are considered in [5]. It is also possible to use the results of [9] to obtain certain analogs for spherical bicentric polygons.

მათემატიკა

ბიცენტრული მრავალკუთხედების შესახებ

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(წარმოდგენილია აკადემიკოს რ. გამყრელიძის მიერ)

ნაშრომში შესწავლილია ბიცენტრული მრავალკუთხედების ერთგანზომილებიანი ოჯახები ფიქსირებული ჩახაზული და შემოხაზული წრეწირებით. კვლევის ძირითადი ობიექტია სახსრული მექანიზმისაგან ინდუცირებული მოდულების სივრცის ტოპოლოგია და ბიცენტრული მრავალ-კუთხედების ექსტრემალური ფართობები. დადგენილია, რომ ბიცენტრული მრავალკუთხედების მოდულების სივრცეს აქვს კვადრატული ტიპის განსაკუთრებული წერტილები, რომელთა რაოდენობა შეფასებულია ზევიდან წრეწირების რადიუსებითა და ცენტრებს შორის მანძილით. დადგენილია შესაბამისი მოდულების სივრცის ეილერის მახასიათებლის ყველა ის შესაძლო მნიშნელობა, რომელსაც ის მიიღებს ბიცენტრული მრავალკუთხედების ოჯახის ცვლილების დროს. აგრეთვე შეფასებულია ბიცენტრული მრავალკუთხედების ოჯახზე განსაზღვრული მრავალკუთხედის ფართობის ფუნქციის კრიტიკული წერტილების რაოდენობა და მიღებულია ექსტრემალური კონფიგურაციების აღწერა. სამკუთხედების შემთხვევაში გამოთვლილია საშუალო ფართობი. ნაშრომში გამოთქმულია აგრეთვე ორი ჰიპოთეზა.

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