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Mathematics

Problem of Elasticity and Plasticity for a Plate with a Shape of *n*-Angle Weakened by *n*-Holes

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ABSTRACT. We consider a homogeneous, isotropic plate with a shape of rectilinear *n*-angle weakened by *n*-cyclic symmetric holes. The plate is in a stressed state; a region of plasticity contains only contours of holes and does not spread inside of the plate. A problem of elasticity and plasticity for this plate is reduced to a boundary problem of linear relationship for a unit circle with sectionally constant coefficients. The equation of unknown contours of holes is presented; the solution of this problem is obtained.

Key words: *stressed condition, region of plasticity, boundary problem of linear relationship for unit circle, unknown part of the boundary.*

Let us consider homogeneous, isotropic plate with a shape of a rectilinear *n*-angle weakened by *n*-similar holes. The contours of the holes are symmetric with respect to radii.

Denote by *S* the region of complex plane $z = x + iy$, occupied by plate. We suppose without limiting the generality that the center of a polygon is in the origin of coordinates and one of the radii is on the OX axis. Let L_0 be outer and $L = \bigcup_{k=1}^{n} L_k$ inner boundaries. Here L_k (k=1, 2, ...,n) is a smooth contour of the k^{th} hole. Assume that the region *S* is cyclic symmetric to the origin of coordinates by a cyclic symmetric angle $\beta = \frac{2\pi}{n}$.

Let us consider the following problem: determine the stressed state of the plate and a shape of contours of holes [1], when the normal displacement on the boundary L_0 is constant and the tangent stress is zero.

$$u_n = const, \ \ \tau_{tn} = 0 \, , \ t \in L_0 \, . \tag{1}$$

Contours of the holes are under the normal stress and the tangent stress is zero too.

$$\sigma_n = -p, \ \ \tau_{tn} = 0 \, , \ t \in L \, . \tag{2}$$

Contour *L* is in a plastic state, which does not spread inside the plate [2-4]:

$$\left(\sigma_t - \sigma_n\right)^2 + 4\tau_{tn}^2 = 4k^2, t \in L. \tag{3}$$

where σ_t is tangential normal stress. Suppose that L_k (k=1; … n) are smooth curves.

On the basis of Kolosov-Muskhelishvili formulae [5] we obtain the following:

$$\sigma_n + i\tau_{tn} = \Phi(z) + \overline{\Phi(z)} - e^{-2i\alpha(t)}\left(z\overline{\Phi'(z)} + \overline{\Psi(z)}\right), \tag{4}$$

$$2\mu\left(u_t' - iu_n'\right) = \varkappa\Phi(z) - \overline{\Phi(z)} + e^{-2i\alpha(t)}\left(z\overline{\Phi'(z)} + \overline{\Psi(z)}\right), \tag{5}$$

where $\Phi(z) = \varphi'(z)$, $\Psi(z) = \psi'(z)$. $\varphi(z)$ and $\psi(z)$ are holomorphic functions in region S. $\alpha(t)$ is the angle between axis OX and an outer normal of the contour at the point *t*.

Because of cyclic symmetry of the problem the normal displacement and tangent stress on apothems and radii of the polygon will be zero, therefore it is sufficient to consider only the shading part of region *S* (see Figure). Denote this part by *D*.

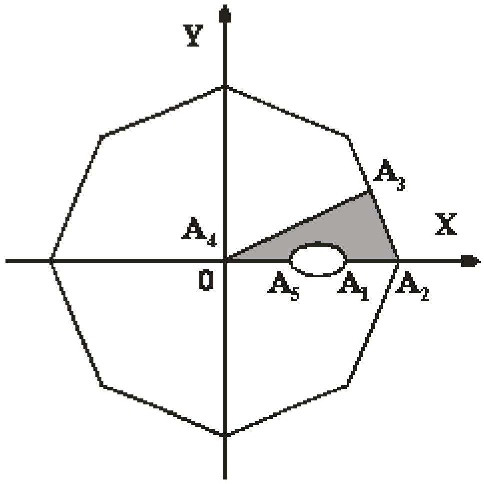

Fig. Cyclic symmetric polygon.

From (1)-(5) it follows that the complex potentials satisfy the conditions:

$$\Phi(z) = a,\ z \in S,\ a = \frac{k-p}{2}, \tag{6}$$

$$e^{2i\alpha(t)}\Psi(t) = b\ ,\ t \in L_1',\ b = k\,, \tag{7}$$

$$\operatorname{Im} e^{2i\alpha(t)}\Psi(t) = 0\ ,\ t \in L_2, \tag{8}$$

where L_1' is a part of L_1 and L_2 is the broken line $A_1A_2A_3A_4A_5$ (see Fig.). Function $\alpha(t)$ is sectionally constant on L_2 and unknown continuous on L_1'. Function $\Psi(z)$ is continuously extendable to the boundary of region *D*, except may be point A_2, where the following inequality holds

$$\left|\Psi(z)\right| < \frac{const}{\left|z - A_2\right|^{\varepsilon}},\ 0 \le \varepsilon < 1\,. \tag{9}$$

In addition to the equalities (7) and (8) let us consider the equation of contour L_2:

$$\operatorname{Re}\left(t \cdot e^{-i\alpha(t)}\right) = \operatorname{Re}\left(A(t) \cdot e^{-i\alpha(t)}\right). \tag{10}$$

$A(t)$ is sectionally constant function and has the form

$A(t) = A_k$, $t \in A_k A_{k+1}$, $k = 1,2,3,4$.

Denote the unit half-circle of the complex ς -plane by D_1, where.

$$D_1 = \{\varsigma : |\varsigma| < 1, \operatorname{Im}\varsigma > 0\}$$

Let function $z = \omega(\varsigma)$ conformally map region D of the z plane onto region D_1 of the ς plane; let a_k be the images of points A_k. Without loss of the generality we suppose that

$a_1 = 1$, $a_3 = i$, $a_5 = -1$.

(7), (8), (10) equalities turn into the form

$$e^{2i\alpha_0(\sigma)}\Psi_0(\sigma) = b, \qquad \sigma \in l_1; \tag{11}$$

$$\operatorname{Im} e^{2i\alpha_0(\sigma)}\Psi_0(\sigma) = 0, \quad \sigma \in l_2; \tag{12}$$

$$\operatorname{Re}\left(e^{-i\alpha_0(\sigma)}\omega(\sigma)\right) = \operatorname{Re}\left(e^{-i\alpha_0(\sigma)}A_0(\sigma)\right), \quad \sigma \in l_2, \tag{13}$$

where l_1 and l_2 are images of L_1' and L_2 contours, respectively;

$\Psi_0(\varsigma) = \Psi(\omega(\varsigma))$, $\alpha_0(\sigma) = \alpha(\omega(\sigma))$, $A_0(\sigma) = A(\omega(\sigma)) = A_k$. $\alpha_0(\sigma)$ is a known sectionally constant function on contour l_2 and unknown on contour l_1, because contour L_1 itself is unknown.

At the conformal mapping $z = \omega(\varsigma)$ expression $e^{2i\alpha_0(\sigma)}$ will have the following form [5]:

$$e^{2i\alpha_0(\sigma)} = \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}, \quad \sigma \in l_1. \tag{14}$$

Taking into account (14) in equation (11) and differentiating (13) we get

$$\overline{\omega'(\sigma)} \cdot \overline{\Psi_0(\sigma)} = b\omega'(\sigma), \ \sigma \in l_1; \tag{15}$$

$$\operatorname{Im} e^{-i\alpha_0(\sigma)}\sigma\omega'(\sigma) = 0, \sigma \in l_2; \tag{16}$$

$$\operatorname{Im} e^{2i\alpha_0(\sigma)}\Psi_0(\sigma) = 0, \sigma \in l_2. \tag{17}$$

The task is reduced to the problem of finding unknown analytical functions $\Psi_0(\varsigma)$ and $\omega(\varsigma)$ in unit semicircle.

Let us introduce the function

$$W(\varsigma) = \begin{cases} b\omega'(\varsigma), & |\varsigma| < 1, \ \operatorname{Im}\varsigma > 0 \\ \overline{\omega'(\overline{\varsigma})} \cdot \overline{\Psi_0(\overline{\varsigma})}, & |\varsigma| < 1, \ \operatorname{Im}\varsigma < 0. \end{cases} \tag{18}$$

From (15) and (18) we have

$$W^+(\sigma) = W^-(\sigma), \quad \sigma \in l_1$$

and, consequently, the function $W(\varsigma)$ is analytic in the unit circle $|\varsigma| < 1$.

Using formulae (16), (17), (18) we obtain

$$\operatorname{Im} e^{-i\alpha_0(\sigma)}\sigma W^+(\sigma) = 0, \quad \sigma \in l, \tag{19}$$

where l is a circumference $|\varsigma| = 1$.

Rewrite equality (19) as follows

$$e^{-i\alpha_0(\sigma)}\sigma\ W^{+}(\sigma)=e^{i\alpha_0(\sigma)}\overline{\sigma W^{+}(\sigma)},\quad \sigma\in l, \tag{20}$$

Define the function

$$F(\varsigma)=\begin{cases}\varsigma\cdot W(\varsigma), & |\varsigma|<1\\ \dfrac{1}{\varsigma}\cdot\overline{W\left(\dfrac{1}{\overline{\varsigma}}\right)}, & |\varsigma|>1\end{cases} \tag{21}$$

Function $F(\varsigma)$ is analytic inside and outside curve l and by virtue of (20), taking into account (21), it satisfies the equality

$$F^{+}(\sigma)=e^{2i\alpha_0(\sigma)}F^{-}(\sigma),\quad \sigma\in l\,. \tag{22}$$

Thus our problem comes to boundary problem of linear relationship for unit circle with sectionally constant coefficients [5].

The solution of problem (22), at the same time, is the solution of Riemann-Hilbert problem (19) if the following condition is satisfied [6]:

$$F(\varsigma)=\overline{F\left(\frac{1}{\overline{\varsigma}}\right)}. \tag{23}$$

Find a solution $F(\varsigma)$ such that

$$F(\varsigma)=0,\ \text{when}\ \varsigma=0\,. \tag{24}$$

For the index of problem (22) we have [6]:

$$\varkappa=3\,. \tag{25}$$

Taking into account equalities (23), (24), (25) for the solution $F(\varsigma)$ of problem (22) we will finally get

$$F(\varsigma)=\chi(\varsigma)\cdot\left(c_1\varsigma^2+\overline{c_1}\varsigma\right). \tag{26}$$

$\chi(\varsigma)$ is the canonical solution of the same (22) problem and in this case has the following form:

$$\chi(\varsigma)=e^{\gamma(\varsigma)}\prod_{k=1}^{8}\left(\varsigma-a_k\right)^{\lambda_k},\ \gamma(\varsigma)=\frac{1}{2\pi i}\int_l\frac{\ln G(\sigma)d\sigma}{\sigma-\varsigma}, \tag{27}$$

where $G(\sigma)$ is a coefficient of the boundary problem of linear relationship (22) ($e^{2i\alpha_0(\sigma)}$ in this case), a_k (k=1;2;3;…8) are nodes of contour l.

$$\lambda_1=\lambda_3=\lambda_4=\lambda_5=0\,;\ \lambda_2=-2\,;\ \lambda_6=-1\,;\ \lambda_7=-1\,;\ \lambda_8=1\,.$$

Taking into account (18), (21), (26), we will finally get

$$t=\omega(\sigma)=\frac{1}{b}\int_{-1}^{\sigma}\chi(\varsigma)\left(c_1\varsigma+\overline{c_1}\right)d\varsigma+C, \tag{28}$$

where $\varsigma\in l_1$, i.e. the integration occurs on the valid axis. Equality (28) is the equation of the required contour.

We can calculate constant C in (28) based on the condition that point $a_5=-1$ is an image of point A_5,

$$A_5=\frac{1}{b}\int_{-1}^{-1}\chi(\varsigma)\left(c_1\varsigma+\overline{c_1}\right)d\varsigma+C\,,$$

the integral on the closed line will be equal to zero and therefore we will have $A_5=C$.

For calculation of a constant c_1 we use a condition that point $a_3 = i$ is image of point A_3.

$$A_3 = \frac{1}{b}\int_{-1}^{i} \chi(\varsigma)\left(c_1\varsigma + \overline{c_1}\right)d\varsigma + C\,.$$

Knowing $\omega(\varsigma)$, from equality (18) we can determine $\Psi_0(\varsigma)$ and also $\Psi(z)$, which with function $\Phi(z)$ determines the stressed state of a plate.

მათემატიკა

დრეკად-პლასტიური ამოცანა წესიერი n-კუთხედის ფორმის მქონე ფირფიტისათვის, რომელიც შესუსტებულია n ერთნაირი ხვრელით

ზ. აბაშიძე

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ნაშრომში განხილულია ერთგვაროვანი, იზოტროპული წესიერი n-კუთხედის ფორმის მქონე ფირფიტა, რომელიც შესუსტებულია n ციკლურად სიმეტრიული ერთნაირი ხვრელით.ფირფიტა იმყოფება დაძაბულ მდგომარეობაში, პლასტიური არე მოიცავს მხოლოდ ხვრელების კონტურებს და არ ვრცელდება ფირფიტის სიღრმეში. ამოცანა მიიყვანება წრფივი შეუღლების სასაზღვრო ამოცანამდე ერთეულრადიუსიანი წრისათვის უბან-უბან მუდმივი კოეფიციენტით. მიღებულია ამოცანის ამონახსნი და მოცემულია ხვრელების უცნობი კონტურების განტოლება.

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