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Mathematics

Clark's Representation of Wiener Functionals and Hedging of the Barrier Option

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ABSTRACT. For one functional of Wiener process, which in case of Bachelier's model of financial market is the payoff of Knock-Out Barrier Option, the Clark's integral representation with explicit form of integrand is obtained. This functional represents the product of European Call Option payoff and indicator of some event. It is impossible to use directly the Clark-Ocone's formula because the indicator of event is Malliavin differentiable if and only if probability of this event is equal to zero or one. We use our integral representation of functionals of Wiener process, which slightly generalizes the Clark-Ocone's formula, and obtain the explicit form of integrand. This integrand is the optimal hedging strategy replicating the Knock-Out Barrier Option in case of Bachelier's model. *© 2014 Bull. Georg. Natl. Acad. Sci.*

Key words: *Brownian functionals, Malliavin derivative, Clark's representation, Clark-Ocone's representation.*

1. Introduction and auxiliary results

Consider a Wiener process $w=(w_t)$, $t\in[0,T]$ on the probability space $(\Omega,\Im,P)$. Let $(\Im_t^w)$, $t\in[0,T]$ be the natural filtration generated by the Wiener process. We study the functionals F_T of the Wiener process, i.e., the random variables, which are $\Im_T^w$ -measurable. Moreover, we are interested in square integrable Wiener functionals, which have the Clark's integral representation.

Assume that a functional has the following form:

$$F_T=(w_T-K)^+ I_{\{w_T^*\le B\}}, \tag{1}$$

where $w_T^*=\max\limits_{0\le s\le T} w_s$; K and B are positive constants such that $B\ge K$; I_A is the indicator of event A. Our aim is to obtain the Clark's integral representation of this square integrable functional with explicit form of integrand. Note that $I_{\{w_T^*\le B\}}$ in general has no Malliavin derivative. It is well known that the indicator of event A is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one [1] and it is impossible to use directly Clark-Ocone's formula. Therefore, we use the result proved by us:

Theorem 1. *Suppose that* $g_t = E[F_T | \Im_t^w]$ *is Malliavin differentiable* $(g_t(\cdot) \in D_{2,1})$ *for almost all* $t \in [0,T)$. *Then we have the representation*

$$g_T = F_T = E[F_T] + \int_0^T v_s dw_s, \quad (P\text{-a.s.}), \tag{2}$$

where

$$v_s := \lim_{t \to T} E(D_s g_t \mid \Im_t^w) \text{ in the } L_2([0,T] \times \Omega).$$

Also, for our calculation it is necessary the following.

Lemma. *Joint conditional density* ($t > s$)

$$f_{w_t, w_t^* | w_s = z}(x, y \mid z) = \frac{\partial^2 P\{w_t \le x, w_t^* = \max_{0 \le l \le t} w_l \le y \mid w_s = z\}}{\partial x \partial y}$$

is the following

$$f_{w_t, w_t^* | w_s = z}(x, y \mid z) = \frac{2(2y - x - z)}{(t-s)\sqrt{2\pi(t-s)}} \exp\{-\frac{(2y - x - z)^2}{2(t-s)}\}, \quad x \le y, \ y \ge 0.$$

Proof of Lemma. It is clear that if $x > y$ or $y < 0$, then the conditional joint probability distribution function of w_t and w_t^* given $w_s = z$ is equal to zero. Now suppose that $x \le y$ and $y \ge 0$.

Due to the elementary relations

$$w_t^* = w_s^* \vee \max_{s \le l \le t} w_l \text{ and} \left(u \vee \max_{s \le l \le t} w_l\right) - w_s = (u - w_s) \vee \left(\max_{s \le l \le t} w_l - w_s\right),$$

using the well-known properties of the Wiener process, one can easily see that:

$$\begin{aligned}
& P\{w_t \le x, w_t^* \le y \mid w_s = z, w_s^* = u\} = \\
& = P\{w_t \le x, w_s^* \vee \max_{s < l \le t} w_l \le y \mid w_s = z, w_s^* = u\} = \\
& = P\{w_t - w_s \le x - z, (u \vee \max_{s < l \le t} w_l) - w_s \le y - z \mid w_s = z, w_s^* = u\} = \\
& = P\{w_t - w_s \le x - z, (u - w_s) \vee \max_{s < l \le t}(w_l - w_s) \le y - z \mid w_s = z, w_s^* = u\} = \\
& = P\{w_t - w_s \le x - z, (u - w_s) \le y - z, \max_{s < l \le t}(w_l - w_s) \le y - z \mid w_s = z, w_s^* = u\} = \\
& = P\{w_t - w_s \le x - z, u \le y, \max_{s < l \le t}(w_l - w_s) \le y - z \mid w_s = z, w_s^* = u\} = \\
& = P\{w_t - w_s \le x - z, \max_{s < l \le t}(w_l - w_s) \le y - z \mid w_s = z, w_s^* = u\} = \\
& = P\{w_t - w_s \le x - z, \max_{s < l \le t}(w_l - w_s) \le y - z\}.
\end{aligned} \tag{3}$$

Let us introduce the new Wiener process $\overline{w}_\theta := w_t - w_{t-\theta}$, $\theta \in [0,t]$. It is evident that: $w_t - w_s = \overline{w}_{t-s}$ and

$$\max_{s < l \le t}(w_l - w_s) = \max_{s \le l \le t}(w_l - w_s) = \max_{s \le l \le t} \overline{w}_{l-s} = \max_{0 \le l - s \le t - s} \overline{w}_{l-s} = \overline{w}^*_{l-s}.$$

Using the well-known result [2: 387] from the point of view of joint distribution function of w_t and w_t^* (with $x \le y$):

$$P\{w_t \le x, w_t^* \le y\} = P\{w_t \le x\} - P\{w_t > 2y - x\} =$$
$$= P\{w_t \le x\} - P\{w_t < x - 2y\} = \frac{1}{\sqrt{2\pi t}} \int_{x-2y}^{x} \exp\{-\frac{r^2}{2t}\} dr,$$

from (3) we obtain that

$$P\{w_t \le x, w_t^* \le y \mid w_s = z, w_s^* = u\} = P\{\overline{w}_{t-s} \le x - z, w_{t-s}^* \le y - z\} =$$
$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{x-z-2(y-z)}^{x-z} \exp\{-\frac{r^2}{2(t-s)}\} dr = \frac{1}{\sqrt{2\pi(t-s)}} \int_{x-2y+z}^{x-z} \exp\{-\frac{r^2}{2(t-s)}\} dr.$$

Therefore, using the well-known properties of conditional expectation, we can write

$$P\{w_t \le x, w_t^* \le y \mid w_s = z\} = [E(I_{\{w_t \le x, w_t^* \le y\}} \mid w_s)] |_{w_s = z} =$$
$$= \{E[E(I_{\{w_t \le x, w_t^* \le y\}} \mid w_s, w_s^*) \mid w_s]\} |_{w_s = z} =$$
$$= \{E[E(I_{\{w_t \le x, w_t^* \le y\}} \mid w_s = z, w_s^* = u) |_{z = w_s, u = w_s^*} \mid w_s]\} |_{w_s = z} =$$
$$= \{E[P(w_t \le x, w_t^* \le y \mid w_s = z, w_s^* = u) |_{z = w_s, u = w_s^*} \mid w_s]\} |_{w_s = z} =$$
$$= \{E[\frac{1}{\sqrt{2\pi(t-s)}} \int_{x-2y+z}^{x-z} \exp\{-\frac{r^2}{2(t-s)}\} dr |_{z = w_s, u = w_s^*} \mid w_s]\} |_{w_s = z} =$$
$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{x-2y+z}^{x-z} \exp\{-\frac{r^2}{2(t-s)}\} dr.$$

Hence, we have

$$\frac{\partial^2 P\{w_t \le x, w_t^* \le y \mid w_s = z\}}{\partial x \partial y} = \frac{\partial}{\partial y} \{\frac{1}{\sqrt{2\pi(t-s)}} [\exp\{-\frac{(x-z)^2}{2(t-s)}\} - \exp\{-\frac{(x-2y+z)^2}{2(t-s)}\}]\} =$$
$$= -\frac{1}{\sqrt{2\pi(t-s)}} \exp\{-\frac{(x-2y+z)^2}{2(t-s)}\} \frac{2(x-2y+z)}{t-s} = \frac{2(2y-x-z)}{(t-s)\sqrt{2\pi(t-s)}} \exp\{-\frac{(2y-x-z)^2}{2(t-s)}\},$$

which completes the proof of the Lemma.

2. Clark's representation of functional F_T

Theorem 2. *For the functional* $F_T = (w_T - K)^+ I_{\{w_T^* \le B\}}$ *the following integral representation is fulfilled*:

$$F_T = EF_T - \int_0^T \frac{2(B-K)}{\sqrt{T-s}} \varphi_{0,1}(\frac{B-w_s}{\sqrt{T-s}}) dw_s + \int_0^T \{\Phi_{0,1}(\frac{w_s - K}{\sqrt{T-s}}) - \Phi_{0,1}[\frac{w_s - (2B-K)}{\sqrt{T-s}}]\} dw_s,$$

where

$$\varphi_{0,1}(x) = \frac{1}{\sqrt{2\pi}} \exp\{-x^2/2\} \text{ and } \Phi_{0,1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-r^2/2\} dr.$$

Proof. Using the well-known properties of conditional expectation, according to the Lemma, we have:

$$g_t = E[F_T \mid \Im_t^w] = E[(w_T - K)^+ I_{\{w_T^* \le B\}} \mid w_t] = \{E[(w_T - K)^+ I_{\{w_T^* \le B\}} \mid w_t = z]\}|_{z=w_t} =$$
$$= \{ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x-K)^+ I_{\{y \le B\}} f_{w_T, w_T^* | w_t = z}(x, y \mid z) dxdy\}|_{z=w_t} =$$
$$= [\int_K^B \int_K^y (x-K) \frac{2(2y-x-z)}{(T-t)\sqrt{2\pi(T-t)}} \exp\{-\frac{(2y-x-z)^2}{2(T-t)}\} dxdy]|_{z=w_t} .$$

Further, using an integration by parts formula in the integral with respect to dx, it is not difficult to see that:

$$\int_K^B \int_K^y (x-K) \frac{2(2y-x-z)}{(T-t)} \frac{1}{\sqrt{2\pi(T-t)}} \exp\{-\frac{(2y-x-z)^2}{2(T-t)}\} dxdy =$$
$$= \int_K^B \frac{1}{\sqrt{2\pi(T-t)}} \int_K^y 2(x-K) d(\exp\{-\frac{(2y-x-z)^2}{2(T-t)}\}) dy =$$
$$= \int_K^B \frac{1}{\sqrt{2\pi(T-t)}} 2(x-K) \exp\{-\frac{(2y-x-z)^2}{2(T-t)}\}|_K^y \, dy -$$
$$- \int_K^B \frac{2}{\sqrt{2\pi(T-t)}} \int_K^y \exp\{-\frac{(2y-x-z)^2}{2(T-t)}\} dxdy.$$

Hence,

$$g_t = \int_K^B \frac{2(y-K)}{\sqrt{2\pi(T-t)}} \exp\{-\frac{(y-w_t)^2}{2(T-t)}\} dy - \int_K^B \frac{2}{\sqrt{2\pi(T-t)}} \int_K^y \exp\{-\frac{(2y-x-w_t)^2}{2(T-t)}\} dxdy .$$

From here according to the rule of stochastic differentiation of the ordinary integral we obtain:

$$D_s g_t = \int_K^B \frac{2(y-K)}{\sqrt{2\pi(T-t)}} \frac{y-w_t}{T-t} \exp\{-\frac{(y-w_t)^2}{2(T-t)}\} I_{[0,t]}(s) dy -$$
$$- \int_K^B \frac{2}{\sqrt{2\pi(T-t)}} \int_K^y \frac{2y-x-w_t}{T-t} \exp\{-\frac{(2y-x-w_t)^2}{2(T-t)}\} I_{[0,t]}(s) dxdy := I_1 + I_2 .$$

Further, using again an integration by parts formula in the integral with respect to dy, we have:

$$I_1 = -\int_K^B \frac{2(y-K)}{\sqrt{2\pi(T-t)}} d(\exp\{-\frac{(y-w_t)^2}{2(T-t)}\}) I_{[0,t]}(s) = -[\frac{2(y-K)}{\sqrt{2\pi(T-t)}} \exp\{-\frac{(y-w_t)^2}{2(T-t)}\}]|_K^B \, I_{[0,t]}(s) +$$
$$+ \int_K^B \frac{2}{\sqrt{2\pi(T-t)}} \exp\{-\frac{(y-w_t)^2}{2(T-t)}\} dy I_{[0,t]}(s) = -2(B-K)\varphi_{0,T-t}(B-w_t) I_{[0,t]}(s) +$$
$$+ 2[\Phi_{0,T-t}(B-w_t) - \Phi_{0,T-t}(K-w_t)] I_{[0,t]}(s).$$

Here and below $\varphi_{0,\sigma^2}(\cdot)$ and $\Phi_{0,\sigma^2}(\cdot)$ are the distribution density and distribution function of normal distributed random variable $N(0,\sigma^2)$ with mean 0 and variance σ^2, respectively, i. e.

$$\varphi_{0,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{x^2}{2\sigma^2}\} \quad \text{and} \quad \Phi_{0,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \exp\{-\frac{r^2}{2\sigma^2}\} dr.$$

$$I_2 = -\int_K^B \frac{2}{\sqrt{2\pi(T-t)}} \int_K^y d(\exp\{-\frac{(2y-x-w_t)^2}{2(T-t)}\}) dy I_{[0,t]}(s) = -\int_K^B \frac{2}{\sqrt{2\pi(T-t)}} \exp\{-\frac{(y-w_t)^2}{2(T-t)}\} dy I_{[0,t]}(s) +$$
$$+\int_K^B \frac{2}{\sqrt{2\pi(T-t)}} \exp\{-\frac{(2y-K-w_t)^2}{2(T-t)}\} dy I_{[0,t]}(s) = -2[\Phi_{0,T-t}(B-w_t) - \Phi_{0,T-t}(K-w_t)] I_{[0,t]}(s) +$$
$$+\int_K^B \frac{1}{\sqrt{2\pi(T-t)/4}} \exp\{-\frac{(y-K/2-w_t/2)^2}{2(T-t)/4}\} dy I_{[0,t]}(s) = -2[\Phi_{0,T-t}(B-w_t) - \Phi_{0,T-t}(K-w_t)] I_{[0,t]}(s) +$$
$$+\{\Phi_{0,(T-t)/2}[(2B-K-w_t)/2] - \Phi_{0,(T-t)/2}[(K-w_t)/2]\} I_{[0,t]}(s).$$

Summing up the above given results, due to the elementary relation $cN(a,\sigma^2) = N(ca, c^2\sigma^2)$, we ascertain that:

$$D_s g_t = -2(B-K)\varphi_{0,T-t}(B-w_t) I_{[0,t]}(s) + 2[\Phi_{0,T-t}(B-w_t) - \Phi_{0,T-t}(K-w_t)] I_{[0,t]}(s) -$$
$$-2[\Phi_{0,T-t}(B-w_t) - \Phi_{0,T-t}(K-w_t)] I_{[0,t]}(s) + \{\Phi_{0,(T-t)/2}[(2B-K-w_t)/2] - \Phi_{0,(T-t)/2}[(K-w_t)/2]\} I_{[0,t]}(s) =$$
$$= -2(B-K)\varphi_{0,T-t}(B-w_t) I_{[0,t]}(s) + [\Phi_{0,2(T-t)}(2B-K-w_t) - \Phi_{0,2(T-t)}(K-w_t)] I_{[0,t]}(s)$$

Now let us pass to calculation of conditional mathematical expectation of $D_s g_t$ with respect to $\Im_s^w$. On the one hand, due to the Markov property and the transition probability of the Wiener process, we can write

$$E[\varphi_{0,T-t}(B-w_t) \mid \Im_s^w] = E[\varphi_{0,T-t}(B-w_t) \mid w_s] =$$
$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} \varphi_{0,T-t}(B-x) \exp\{-\frac{(x-w_s)^2}{2(t-s)}\} dx =$$
$$= \frac{1}{\sqrt{2\pi(t-s)}} \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \exp\{-\frac{(B-x)^2}{2(T-t)}\} \exp\{-\frac{(x-w_s)^2}{2(t-s)}\} dx.$$

Hence, highlighting the full square in the argument of the exponential function and using the well-known property of the distribution function, we obtain that

$$E[\varphi_{0,T-t}(B-w_t) \mid \Im_s^w] = E[\varphi_{0,T-t}(B-w_t) \mid w_s] = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} \varphi_{0,T-t}(B-x) \exp\{-\frac{(x-w_s)^2}{2(t-s)}\} dx =$$
$$= \frac{1}{\sqrt{2\pi(t-s)}} \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \exp\{-\frac{T-s}{2(T-t)(t-s)}[x - \frac{B(t-s)+w_s(T-t)}{T-s}]^2 - \frac{(B-w_s)^2}{2(T-s)}\} dx =$$
$$= \frac{1}{\sqrt{2\pi(t-s)}} \frac{1}{\sqrt{2\pi(T-t)}} \exp\{-\frac{(B-w_s)^2}{2(T-s)}\} \int_{-\infty}^{\infty} \exp\{-\frac{T-s}{2(T-t)(t-s)}[x - \frac{B(t-s)+w_s(T-t)}{T-s}]^2\} dx =$$
$$= \frac{1}{\sqrt{2\pi(t-s)}} \frac{1}{\sqrt{2\pi(T-t)}} \exp\{-\frac{(B-w_s)^2}{2(T-s)}\} \sqrt{2\pi \frac{(T-t)(t-s)}{T-s}} =$$
$$= \frac{1}{\sqrt{2\pi(T-s)}} \exp\{-\frac{(B-w_s)^2}{2(T-s)}\} = \varphi_{0,T-s}(B-w_s).$$

On the other hand, using again the Markov property and the transition probability of the Wiener process, we have

$$E[\Phi_{0,2(T-t)}(C-w_t)\,|\,\Im_s^w]=E[\Phi_{w_t,2(T-t)}(C)\,|\,\Im_s^w]=E[\Phi_{w_t,2(T-t)}(C)\,|\,w_s]=$$
$$=\frac{1}{\sqrt{2\pi 2(T-t)}}\frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty}[\int_{-\infty}^{C}\exp\{-\frac{(u-x)^2}{4(T-t)}\}du]\exp\{-\frac{(x-w_s)^2}{2(t-s)}\}dx=$$
$$=\frac{1}{\sqrt{4\pi(T-t)}}\frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}I_{(-\infty,C)}(u)\exp\{-\frac{(u-x)^2}{4(T-t)}\}\exp\{-\frac{(x-w_s)^2}{2(t-s)}\}dudx.$$

Therefore, according to the Fubini's theorem, using the argument similar to those presented above, it is not difficult to see that

$$E\left[\Phi_{0,2(T-t)}(C-w_t)\,|\,\Im_s^w\right]=$$
$$=\frac{1}{\sqrt{4\pi(T-t)}}\frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty}I_{(-\infty,C)}(u)[\int_{-\infty}^{\infty}\exp\{-\frac{(u-x)^2}{4(T-t)}\}\exp\{-\frac{(x-w_s)^2}{2(t-s)}\}dx]du=$$
$$=\int_{-\infty}^{C}\{\frac{1}{\sqrt{4\pi(T-t)}}\frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty}\exp\{-\frac{[x-\frac{u(t-s)+2w_s(T-t)}{2T-t-s}]^2+\frac{2(t-s)(T-t)(u-w_s)^2}{(2T-t-s)^2}}{\frac{4(T-t)(t-s)}{2T-t-s}}\}dx\}du=$$
$$=\int_{-\infty}^{C}\{\frac{1}{\sqrt{4\pi(T-t)}}\frac{1}{\sqrt{2\pi(t-s)}}\exp\{-\frac{(u-w_s)^2}{2(2T-t-s)}\}\int_{-\infty}^{\infty}\exp\{-\frac{[x-\frac{u(t-s)+2w_s(T-t)}{2T-t-s}]^2}{\frac{4(T-t)(t-s)}{2T-t-s}}\}dx\}du=$$
$$=\int_{-\infty}^{C}\{\frac{1}{\sqrt{4\pi(T-t)}}\frac{1}{\sqrt{2\pi(t-s)}}\exp\{-\frac{(u-w_s)^2}{2(2T-t-s)}\}\sqrt{4\pi\frac{(T-t)(t-s)}{2T-t-s}}du=$$
$$=\int_{-\infty}^{C}\{\frac{1}{\sqrt{2\pi(2T-t-s)}}\exp\{-\frac{(u-w_s)^2}{2(2T-t-s)}\}du=$$
$$=\Phi_{0,2T-t-s}(C-w_s).$$

Combining all the above obtained relations we easily conclude that

$$E[D_s g_t\,|\,\Im_s^w]=-2(B-K)\varphi_{0,T-s}(B-w_s)I_{[0,t]}(s)+$$
$$+[\Phi_{0,2T-t-s}(2B-K-w_s)-\Phi_{0,2T-t-s}(K-w_s)]I_{[0,t]}(s).$$

Passing now to the limit in the latter expression at $t\to T$ we obtain that

$$v_s:=\lim_{t\to T}E(D_s g_t\,|\,\Im_s^w)=-2(B-K)\varphi_{0,T-s}(B-w_s)I_{[0,T]}(s)+$$
$$+[\Phi_{0,2T-s}(2B-K-w_s)-\Phi_{0,2T-s}(K-w_s)]I_{[0,T]}(s)=$$
$$=-\frac{2(B-K)}{\sqrt{T-s}}\varphi_{0,1}(\frac{B-w_s}{\sqrt{T-s}})I_{[0,T]}(s)+\{\Phi_{0,1}(\frac{w_s-K}{\sqrt{T-s}})-\Phi_{0,1}[\frac{w_s-(2B-K)}{\sqrt{T-s}}]\}I_{[0,T]}(s),$$

which completes the proof of the Theorem 2.

3. Hedging of the Knock-Out Barrier Option

Consider now Knock-Out Barrier Option in the financial market, represented by Bachelier's model. In this case the stock price S is described by

$$S_t = S_0 + \mu t + \sigma w_t .$$

The payoff of Up-and-Out Call Barrier Option is [3]

$$\overline{F}_T = (S_T - \overline{K})^+ I_{\{\max_{0\le t\le T} S_t \le \overline{B}\}}, \quad \overline{K} \le \overline{B} .$$

Consider the unique martingale (risk neutral) measure $\overline{P} \sim P$ such that [4]

$$d\overline{P} = Z_T dP_T$$

with

$$Z_t = \exp\{-\frac{\mu}{\sigma} w_t - \frac{1}{2}(\frac{\mu}{\sigma})^2 t\} .$$

Under this measure the following is fulfilled:

$$\text{Law}(S_0 + \mu t + \sigma w_t ; t \le T \mid \overline{P}_T) = \text{Law}(S_0 + \sigma w_t ; t \le T \mid P_T)$$

and if $\sigma = 1$

$$E^{\overline{P}}[\overline{F}_T \mid \Im_t^w] = E[F_T \mid \Im_t^w],$$

where F_T is given in (1) with $K = \overline{K} - S_0$ and $B = \overline{B} - S_0$.
Using the previous calculations we obtain that

$$\overline{F}_T = EF_T + \int_0^T \{-\frac{2(B-K)}{\sqrt{T-s}} \varphi_{0,1}(\frac{B-w_s}{\sqrt{T-s}}) + \Phi_{0,1}(\frac{w_s - K}{\sqrt{T-s}}) - \Phi_{0,1}[\frac{w_s - (2B-K)}{\sqrt{T-s}}]\} dw_s ,$$

whose integrand defines the optimal hedging strategy.

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ვინერის პროცესის ერთი ფუნქციონალისათვის, რომელიც ფინანსური ბაზრის ბაშელიეს მოდელის შემთხვევაში წარმოადგენს გადასახადის ფუნქციას ნოკაუტ ბარიერული ოფციონისათვის, მიღებულია კლარკის ინტეგრალური წარმოდგენა ცხადი ინტეგრანდით. ეს ფუნქციონალი წარმოადგენს ევროპული კოლოფციონის გადასახადის ფუნქციისა და გარკვეული ხდომილების ინდიკატორის ნამრავლს. აქ კლარკ-ოკონის ფორმულის პირდაპირი გამოყენება შეუძლებელია, რადგანაც ხდომილების ინდიკატორი არის მალიავინის აზრით წარმოებადი მაშინ და მხოლოდ მაშინ, როდესაც ამ ხდომილების ალბათობა ნული ან ერთია. ვიყენებთ ჩვენს მიერ მიღებულ ვინერის ფუნქციონალების ინტეგრალურ წარმოდგენას, რომელიც წარმოადგენს კლარკ-ოკონის ფორმულის მცირე განზოგადებას და ვიღებთ ინტეგრანდის ცხად ფორმას. ეს ინტეგრანდი არის ოპტიმალური ჰეჯური სტრატეგია ნოკაუტ ბარიერული ოფციონის რეპლიკაციისათვის ბაშელიეს მოდელის შემთხვევაში.

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