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Mathematics

Nonlocal Contact Problem for Two-Dimensional Linear Elliptic Equations

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ABSTRACT. A nonlocal contact boundary problem for two-dimensional linear elliptic equations is stated and investigated. The uniqueness of the solution is proved. The iteration process is constructed, which allows one not only to prove the existence of a regular solution of the problem, but also to develop an approximate algorithm of its solution. The solution of a nonlocal contact problem is reduced to the solution of classical boundary value problems, in particular to the solution of Dirichlet problems.
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Key words: *elliptic equation, nonlocal problem, contact problem.*

1. Introduction. The history of investigation of nonlocal boundary value problems began in the 20th century [1,2]. The publication of [3, 4] gave impetus to numerous studies in this direction. The results obtained in these works were generalized and refined from the standpoints of theory and application [5-19].

In the last two decades, extensive studies of nonlocal initial-boundary and boundary value problems were carried out, general theoretical fundamental principles of analysis were formulated, methods were developed for the numerical solution of problems and for the construction of mathematical models of concrete problems in physics, ecology, biology, economics and other areas.

In the present paper, the boundary value problem with nonlocal contact conditions for linear elliptic equations of the second order is stated and investigated in two-dimensional domains. The iteration process is constructed, which allows one to reduce the solution of the initial problem to the solution of a sequence of classical Dirichlet problems. It is obvious that the investigation of nonlocal contact problems is an important direction of applied and computational mathematics.

2. Statement of the Problem. Let us consider the bounded domain D in R^2 with piecewise smooth boundary Γ. We choose two simple points A_0, B_0 [18] on Γ and assume that at these points the tangent to Γ exists. Further, we draw, in D the simple smooth curve Γ_0, connecting A_0 and B_0. It is assumed, that the curve Γ_0 has the tangents at A_0 and B_0, not coinciding with the tangents of the contour Γ in the same points. It is obvious that Γ_0 divides D into two parts (domains) D^- and D^+, and the boundary Γ into two curves Γ_1 and Γ_2 so that $D = D^- \cup D^+ \cup \Gamma_0$, $\bar{D}^- = \Gamma_1 \cup \Gamma_0 \cup D^-$, $\bar{D}^+ = \Gamma_2 \cup \Gamma_0 \cup D^+$ (Fig.1).

Assume that Γ_- is the diffeomorphic image of Γ_0, which lies in D^- and adjoins Γ_1 at the points A_- and B_-; Γ_- and Γ_1 are not the tangents at the points A_- and B_-.

Assume also that Γ_+ is the diffeomorphic image of Γ_0, which lies in D^+ and adjoins Γ_2 at the points A_+ and B_+; Γ_+ and Γ_2 are not the tangents at the points A_+ and B_+.

Assume further that the point A_- is between the points A_0 and B_-, and the point B_- is between the points A_- and B_0. Also assume that the point A_+ is between the points A_0 and B_+, and the point B_+ is between the points A_+ and B_0. The points are assumed to be positioned along the curves Γ_1 and Γ_2, respectively. Also, it is assumed that $\Gamma_0 \cap \Gamma_- = Æ$, $\Gamma_0 \cap \Gamma_+ = Æ$ and the distance between $\Gamma_0, \Gamma_-, \Gamma_+$ is greater than some positive number $\varepsilon = const > 0$.

Let us introduce the notation: $\mu^-(\Gamma_0) = \Gamma_-$, $\mu^+(\Gamma_0) = \Gamma_+$, where $\mu^-(\cdot)$ and $\mu^+(\cdot)$ are the diffeomorphisms between Γ_0 and Γ_-, Γ_0 and Γ_+.

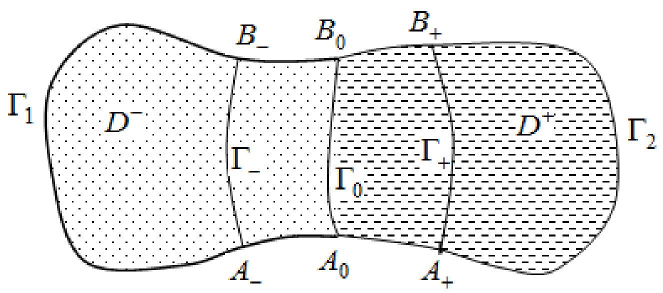

Fig. Domain D

In the domain $\bar{D}$ we consider the problem: find in $\bar{D}$ a continuous function $u(x,y)$,

$$u(x,y) = \begin{cases} u^-(x,y), & if\ (x,y) \in \bar{D}^-, \\ u_0(x,y), & if\ (x,y) \in \Gamma_0, \\ u^+(x,y), & if\ (x,y) \in \bar{D}^+, \end{cases}$$

which satisfies the equations

$$L^- u^- = f^-, \quad if\ (x,y) \in D^-, \tag{1}$$

$$L^+ u^+ = f^+, \quad if\ (x,y) \in D^+, \tag{2}$$

where L^- and L^+ are the second order linear uniformly elliptic operators, f^- and f^+ are given, sufficiently smooth functions in $\bar{D}^-$ and $\bar{D}^+$, respectively.

The function $u(x,y)$ also satisfies the boundary conditions

$$u^-(x,y) = \varphi^-(x,y), \quad if\ (x,y) \in \Gamma_1, \tag{3}$$

$$u^+(x,y) = \varphi^+(x,y), \quad if\ (x,y) \in \Gamma_2, \tag{4}$$

the nonlocal contact conditions

$$u^{-}(\Gamma_0)=u^{+}(\Gamma_0), \tag{5}$$

$$u^{\pm}(\Gamma_0)=u(\Gamma_0)=\gamma^{+}u^{+}(\Gamma_{+})+\gamma^{-}u^{-}(\Gamma_{-})+\varphi_0(\Gamma_0), \tag{6}$$

and the coordination conditions

$$u(B_0)=\gamma^{+}u(B_{+})+\gamma^{-}u(B_{-})+\varphi_0(B_0), \tag{7}$$

$$u(A_0)=\gamma^{+}u(A_{+})+\gamma^{-}u(A_{-})+\varphi_0(A_0), \tag{8}$$

where $\gamma^{-}=const\ge 0$, $\gamma^{+}=const\ge 0$, φ_0,φ^{-} and φ^{+} are known continuous functions.

Assume that the following conditions are satisfied:

I. f^{-}, f^{+} are functions such that for any continuous function $\bar{\varphi}^{-}$ and $\bar{\varphi}^{+}$ in the domains D^{-} and D^{+} a unique regular solution of the problems exists

$$\begin{aligned}&L^{-}v^{-}=f^{-}, \quad (x,y)\in D^{-},\\ &v^{-}=\bar{\varphi}^{-}, \quad (x,y)\in\Gamma_1\cup\Gamma_0\end{aligned} \quad \text{and} \quad \begin{aligned}&L^{+}v^{+}=f^{+}, \quad (x,y)\in \overset{,}{D}{}^{+},\\ &v^{+}=\bar{\varphi}^{+}, \quad (x,y)\in\Gamma_2\cup\Gamma_0;\end{aligned}$$

II. If we have the iteration process

$$\begin{aligned}&L^{-}v^{-(k)}=f^{-}, \quad (x,y)\in D^{-},\\ &v^{-(k)}=\bar{\varphi}^{-(k)}, \quad (x,y)\in\Gamma_1\cup\Gamma_0\end{aligned} \quad \text{and} \quad \begin{aligned}&L^{+}v^{+(k)}=f^{+}, \quad (x,y)\in \overset{,}{D}{}^{+},\\ &v^{+(k)}=\bar{\varphi}^{+(k)}, \quad (x,y)\in\Gamma_2\cup\Gamma_0,\end{aligned}$$

where $f^{\pm}$, $\bar{\varphi}^{\pm(k)}$ are the prescribed functions, $\bar{\varphi}^{\pm(k)}$ uniformly tends to $\bar{\varphi}^{\pm}$, then we have $v^{\pm}(x,y)=\lim\limits_{k\to\infty}v^{\pm(k)}$, which is the regular solution of the problems

$$\begin{aligned}&L^{-}v^{-}=f^{-}, \quad (x,y)\in D^{-},\\ &v^{-}=\bar{\varphi}^{-}, \quad (x,y)\in\Gamma_1\cup\Gamma_0\end{aligned} \quad \text{and} \quad \begin{aligned}&L^{+}v^{+}=f^{+}, \quad (x,y)\in \overset{,}{D}{}^{+},\\ &v^{+}=\bar{\varphi}^{+}, \quad (x,y)\in\Gamma_2\cup\Gamma_0,\end{aligned}$$

this is the generalized Harnak's first theorem [19-21].

III. Assume that the functions $u^{\pm}(x,y)$ are solutions of the equations

$$L^{-}u^{-}=0, \quad (x,y)\in D^{-}, L^{+}u^{+}=0, \quad (x,y)\in D^{+},$$

Schwarz' Lemma is valid for them [18-20].

IV. $\gamma^{+}=const\ge 0$, $\gamma^{-}=const\ge 0$, $\gamma^{-}+\gamma^{+}\le 1$.

V. L^{-} and L^{+} are operators such that the extremum principle [19-22] holds for regular solutions of the equations $L^{-}v^{-}=0$ in D^{-} and $L^{+}v^{+}=0$ in $\overset{,}{D}{}^{+}$.

3. Uniqueness of solution of the problem (1)-(8). The following theorem is true

Theorem 1. If the regular solution of the problem (1)-(8) exists and conditions IV, V are fulfilled, then the solution is unique.

Proof. Suppose that the problem (1)-(8) has two solutions: $v(x,y)$ and $w(x,y)$. Then for the function $z(x,y)=v(x,y)-w(x,y)$ we have the following problem:

$$L^{-}z^{-}=0,\quad if\ (x,y)\in D^{-},\tag{9}$$

$$L^{+}z^{+}=0,\quad if\ (x,y)\in D^{+},\tag{10}$$

$$z^{-}\big|_{\Gamma_1}=0,\ z^{+}\big|_{\Gamma_2}=0,\tag{11}$$

$$z(\Gamma_0)=z^{\pm}(\Gamma_0)=\gamma^{+}z^{+}(\Gamma_{+})+\gamma^{-}z^{-}(\Gamma_{-}),\tag{12}$$

From equality (9)-(12) it follows that

$$\max\left|z(\Gamma_0)\right|\le\gamma^{+}\max\left|z^{+}(\Gamma_{+})\right|+\gamma^{-}\max\left|z^{-}(\Gamma_{-})\right|.$$

Taking into account the condition $\gamma^{-}+\gamma^{+}\le 1$, we obtain

$$\max\left|z(\Gamma_0)\right|\le\max\left|z^{+}(\Gamma_{+})\right|\quad or\quad \max\left|z(\Gamma_0)\right|\le\max\left|z^{-}(\Gamma_{-})\right|.$$

This means that the function z does not attain a maximum on Γ_0, but according to condition V it attains a maximum on Γ_1 or Γ_2. Taking condition (11) into account, we easily obtain $z\equiv 0$, i.e., the solution of the problem (1)-(8) is unique.

4. Existence of the solution of the problem (1)-(8). We consider the following iteration process:

$$L^{-}\left(u^{-}\right)^{(k)}=f^{-},\quad if\ (x,y)\in D^{-},\tag{13}$$

$$L^{+}\left(u^{+}\right)^{(k)}=f^{+},\quad if\ (x,y)\in D^{+},\tag{14}$$

$$\left(u^{-}\right)^{(k)}=\varphi^{-},\quad if\ (x,y)\in\Gamma_1,\tag{15}$$

$$\left(u^{+}\right)^{(k)}=\varphi^{+},\quad if\ (x,y)\in\Gamma_2,\tag{16}$$

$$u^{(k)}(\Gamma_0)=\gamma^{+}u^{+(k-1)}(\Gamma_{+})+\gamma^{-}u^{-(k-1)}(\Gamma_{-})+\varphi_0(\Gamma_0),\tag{17}$$

where $k=0,1,2,\ldots$ and $\left(u^{-}\right)^{(-1)}(\Gamma_{-})=0$, $\left(u^{+}\right)^{(-1)}(\Gamma_{+})=0$, $\Gamma_{+}=\mu^{+}(\Gamma_0),\quad \Gamma_{-}=\mu^{-}(\Gamma_0)$.

Denote

$$\left(z^{\pm}\right)^{(k)}=z^{\pm(k)}(x,y)=z^{\pm(k)}(x,y)-z^{\pm}(x,y),$$

then for the function $z(x,y)$ we obtain the problem

$$L^{-}\left(z^{-}\right)^{(k)}=0,\quad if\ (x,y)\in D^{-},\tag{18}$$

$$L^{+}\left(z^{+}\right)^{(k)}=0,\quad if\ (x,y)\in D^{+},\tag{19}$$

$$\left(z^{-}\right)^{(k)}=0,\quad if\ (x,y)\in\Gamma_1,\ \left(z^{+}\right)^{(k)}=0,\quad if\ (x,y)\in\Gamma_2,\tag{20}$$

and the nonlocal contact condition

$$\left(z^{+}\right)^{(k)}=0,\quad if\ (x,y)\in\Gamma_2,\tag{21}$$

Let L_{-} and L_{+} be the elliptic operators, for which condition III is true. Then from (18)-(21) we get

$$\max_{\Gamma_+}\left|\left(z^+\right)^{(k)}\right| \le q^+ \max_{\Gamma_0}\left|z^{(k)}\right| \quad or \quad \max_{\Gamma_-}\left|\left(z^-\right)^{(k)}\right| \le q^- \max_{\Gamma_0}\left|z^{(k)}\right|,$$

where $q^+ = const$, $0 < q^+ < 1$, $q^- = const$, $0 < q^- < 1$ (q^+ and q^- are the coefficients contained in Schwarz' lemma).

If we use the nonlocal contact condition (21), then we have

$$\max\left|z^{(k)}\left(\Gamma_0\right)\right| \le \left[\gamma^+ q^+ + \gamma^- q^-\right]\max\left|z^{(k-1)}\left(\Gamma_0\right)\right|$$

or

$$\max\left|z^{(k)}\left(\Gamma_0\right)\right| \le Q\max\left|z^{(k-1)}\left(\Gamma_0\right)\right|, \tag{22}$$

where

$$Q = \gamma^+ q^+ + \gamma^- q^- .$$

Taking condition (4) into account, we obtain $0 < Q < 1$. This implies that

$$\lim_{k\to\infty} z^{(k)}\left(\Gamma_0\right) = 0 .$$

If the solution of the problem (1)-(8) exists, then by the maximum principle we obtain

$$\max_{\bar{D}_-}\left|u^{-(k)}\left(x,y\right) - u^-\left(x,y\right)\right| = O\left(Q^k\right),$$

$$\max_{\bar{D}_+}\left|u^{+(k)}\left(x,y\right) - u^+\left(x,y\right)\right| = O\left(Q^k\right),$$

and, accordingly,

$$\max_{\bar{D}}\left|u^{(k)}\left(x,y\right) - u\left(x,y\right)\right| = O\left(Q^k\right).$$

Thereby we proved the following theorem.

Theorem 2. If the solution of problem (1)-(8) exists and condition III is fulfilled, then the iteration process (13)-(17) converges to this solution at the rate of an infinitely decreasing geometric progression.

Theorem 3. If conditions I-V are satisfied, then there exists a regular solution of problem (1) - (8).

Let us now prove the existence of a regular solution of the problem (1)-(8). We introduce the notation $\varepsilon^{(k)}\left(x,y\right) = u^{(k)}\left(x,y\right) - u^{(k-1)}\left(x,y\right)$. Then for the function $\varepsilon^{(k)}$ we obtain the problem

$$L^-\varepsilon^{-(k)} = 0, \quad if\ \left(x,y\right)\in D^-,$$

$$L^+\varepsilon^{+(k)} = 0, \quad if\ \left(x,y\right)\in D^+,$$

$$\varepsilon^{-(k)} = 0, \quad if\ \left(x,y\right)\in\Gamma_1,\ \varepsilon^{+(k)} = 0, \quad if\ \left(x,y\right)\in\Gamma_2,$$

$$\varepsilon^{(k)}\left(\Gamma_0\right) = \varepsilon^{\pm(k)}\left(\Gamma_0\right) = \gamma^+\varepsilon^{+(k-1)}\left(\Gamma_+\right) + \gamma^-\varepsilon^{-(k-1)}\left(\Gamma_-\right),$$

where $k = 0,1,2,\ldots$ and $\left(\varepsilon^-\right)^{(-1)}\left(\Gamma_-\right) = 0$, $\left(\varepsilon^+\right)^{(-1)}\left(\Gamma_+\right) = 0$.

Then, analogously to (22), we obtain the estimate

$$\max\left|\varepsilon^{(k)}\left(\Gamma_0\right)\right| \le Q\max\left|\varepsilon^{(k-1)}\left(\Gamma_0\right)\right|, \quad 0 < Q < 1,$$

or

$$u^{(k)}\left(x,y\right) - u^{(k-1)}\left(x,y\right) \to 0, \quad if \quad k\to\infty \quad and \quad \left(x,y\right)\in\Gamma_0 .$$

This means that the sequence $\left\{u^{(k)}(x,y)\right\}$ converges uniformly on Γ_0. Then for the domains D^- and D^+ we obtain the sequence, which satisfies equations (13), (14) and equalities (15)-(17). From this and condition III we conclude that the limit function is the regular solution of problem (1)-(8):

$$\lim_{k\to\infty} u^{(k)}(x,y) = u(x,y).$$

We have thereby proved that by using the iteration algorithm the solution of a nonclassical contact problem is reduced to the solution of the sequence of classical Dirichlet problems..

Remark. If $\gamma^- + \gamma^+ < 1$ and conditions I, II and V are satisfied, then it is obvious that Theorems 2 and 3 are valid in that case, too.

მათემატიკა

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წარმოდგენილ ნაშრომში ორგანზომილებიანი წრფივი ელიფსური ტიპის განტოლებებისათვის დასმულია და ამოხსნილია არალოკალური სასაზღვრო საკონტაქტო ამოცანა. დამტკიცებულია ამ ამოცანის ამოხსნის ერთადერთობა; აგებულია იტერაციული პროცესი, რომელიც საშუალებას იძლევა დავამტკიცოთ არა მარტო დასმული ამოცანის რეგულარული ამონახსნის არსებობა, არამედ ავაგოთ მისი მიახლოებითი ამოხსნის ალგორითმი. არალოკალური საკონტაქტო ამოცანის ამოხსნა დაიყვანება კლასიკური სასაზღვრო ამოცანების, კერძოდ, დირიხლეს ამოცანების ამოხსნაზე.

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