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Mathematics

On the Lower Level Sets and Anti Fuzzy Ternary Polygroups

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ABSTRACT. The notion of ternary polygroup is a generalization of the notion of polygroup in the sense of Comer. In this paper, we study the concepts of fuzzy and anti fuzzy ternary subpolygroups of a ternary polygroup and discuss some properties of them. *© 2014 Bull. Georg. Natl. Acad. Sci.*

Key words: *hypergroup, polygroup, ternary polygroup, fuzzy set.*

Introduction

Fuzzy sets are sets whose elements have degrees of membership. Fuzzy sets have been introduced by Zadeh [1] as an extension of the classical notion of set. A classical set (ordinary set) is defined by crisp boundaries,i.e., there is no uncertainty in the prescription or location of the boundaries of the set.On the other hand, a fuzzy set, is prescribed by vague orambiguous properties; hence, its boundaries are ambiguously specified. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition - an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. In our daily life , we usually want to seek opinions from professional persons with the best qualifications, for example, the best medical doctors can provide the best diagnostics, the best pilots can provide the best navigation suggestions for airplanes etc. It is therefore desirable to incorporate the knowledge of these experts into some automatic systems so that it would become helpful for other people to make appropriate decisions, which are (almost) as good as the decisions made by the top experts. With this aim in mind, our task is to design a system that would provide the best advice from the best experts in the field. However, one of the main hurdles of this incorporation is that the experts are usually unable to describe their knowledge by using precise and exact terms. For example, in order to describe the size of certain type of a tumor, a medical doctor would rarely use the exact numbers. Instead he would say something like “the size is between 1.4 and 1.6 cm“. Also, an expert would usually use some words from a natural language, e.g., “the size of the tumor is approximately 1.5 cm, with an error of about 0.1 cm”. Thus, under such circumstances, the way to formalize the statements given by an expert is one of the main objectives of fuzzy logic.

Definition 1. Let X be a set. A *fuzzy subset* A of X is characterized by a membership function $\mu_A : X \to [0,1]$, which associates with each point $x \in X$ its grade or degree of membership $\mu_A(x) \in [0,1]$.

Definition 2. Let A and B be two fuzzy subsets of X. Then

(1) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$, for all $x \in X$,

(2) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$,

(3) $C = A \cup B$ if and only if $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$,

(4) $D = A \cap B$ if and only if $\mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$,

(5) The *complement* of A, denoted by A^c, is defined by $\mu_{A^c}(x) = 1 - \mu_A(x)$, for all $x \in X$.

Notice that when the range of membership functions is restricted to the set $\{0,1\}$, these functions perform precisely as the corresponding operators for crisp subsets. For the sake of simplicity, we shall show every fuzzy subset by its membership function.

After the introduction of fuzzy sets by Zadeh [1], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroup was defined by Rosenfeld [2] and its structure was investigated. This subject has been studied further by many others. Das [3] characterized fuzzy subgroups by their level subgroups, since then many notions of fuzzy group theory can be equivalently characterized with the help of notion of level subgroups.The concept of anti fuzzy subgroup was given in [4]. Davvaz applied fuzzy sets to the theory of algebraic hyperstructures [5] and defined the concept of fuzzy subhypergroup of a hypergroup.

The theory of algebraic hyperstructures which is a generalization of the concept of algebraic structures first was introduced by Marty [6], and had been studied in the following decades and nowadays by many mathematicians, and many papers concerning various hyperstructures have appeared in the literature. The basic definitions of the object can be found in [7,8]. The concept of n-ary hypergroup is defined in [9], which is a generalization of the concept of hypergroup in the sense of Marty and a generalization of n-ary group, too. In [10] introduced the notion of a fuzzy n-ary subhypergroup of an n-ary hypergroup. Then this concept studied in [11-18]. A ternary hypergroup is a particular case of an n-ary hypergroup for n=3. Davvaz and Leoreanu-Fotea [19] studied the ternary hypergroups associated with a binary relations. In [20], Davvaz et al. provided examples of ternary hyperstructures associated with chain reactions in chemistry.

In this paper, we study the concepts of fuzzy and anti fuzzy ternary subpolygroups of a ternary polygroup and we discuss some properties of them.

Polygroups and Ternary Polygroups

A *hypergroupoid* is a set H together with a *hyperoperation* $*$ from $H \times H$ into the family of non-empty subsets of H. The image of the pair (x, y) is denoted by $x * y$. If $x \in H$ and A ,B be non-empty subsets of H, then by $A * B$, $A * x$ and $x * B$ we mean

$$A * B = \bigcup_{\substack{a \in A \\ b \in B}} a * b, \quad A * x = A * \{x\}, \quad x * B = \{x\} * B.$$

A hypergroupoid $(H,*)$ is called a *semihypergroup* if $(x*y)*z = x*(y*z)$ for all $x, y, z \in H$.

A hyperstructure $(H,*)$ is called a *hypergroup* if the following axioms hold:

(1) $(x*y)*z = x*(y*z)$ for all $x, y, z \in H$,

(2) $a*H = H*a = H$ for all $a \in H$.

Davvaz [5] applied fuzzy sets to the theory of algebraic hyperstructures and studied their fundamental properties.

Definition 3. Let $(H,*)$ be a hypergroup and let μ a fuzzy subset of *H*. Then, μ is said to be a *fuzzy subhypergroup* of *H* if the following axioms hold:

(1) $\min\{\mu(x), \mu(y)\} \le \inf_{z \in x*y} \{\mu(z)\}$ for all $x, y \in H$,

(2) for all $x, a \in H$ there exists $y \in H$ such that $x \in a*y$ and $\min\{\mu(a), \mu(x)\} \le \mu(y)$,

(3) for all $x, a \in H$ there exists $z \in H$ such that $x \in z*a$ and $\min\{\mu(a), \mu(x)\} \le \mu(z)$.

Application of hypergroups have mainly appeared in special subclasses. For example, polygroups which are certain subclasses of hypergroups are studied by Comer [21] and are used to study color algebra. Quasi-canonical hypergroups (called "polygroups" by Comer) were introduced in [22], as a generalization of canonical hypergroups introduced by Mittas [23]. Some algebraic and combinatorial properties were developed by Comer. We recall the following definition from Comer [21].

Definition 4. A *polygroup* is a multi-valued system $\langle P, *, e, ^{-1} \rangle$ where $^{-1}: P \to P$, and * is a hyperoperation from $P \times P$ into the family of non-empty subsets of *P* such that the following axioms hold:

(1) $(x*y)*z = x*(y*z)$ for all $x, y, z \in P$,

(2) $e*x = x*e = x$,

(3) $x \in y*z$ implies $y \in x*z^{-1}$ and $z \in y^{-1}*x$.

Zahedi et al. [24] defined the concept of fuzzy subpolygroups of a polygroup which is a generalization of the concept of Rosenfeld's fuzzy subgroups and special case of Davvaz's definition for fuzzy subhypergroups

Definition 5. Let $\langle P, *, e, ^{-1} \rangle$ be a polygroup and let μ a fuzzy subset of *P*. Then μ is said to be a *fuzzy subpolygroup* of *P* if the following axioms hold:

(1) $\min\{\mu(x), \mu(y)\} \le \mu(z)$ for all $x, y \in P$ and for all $z \in x*y$,

(2) $z \in x*y$, for all $x \in P$.

Let *H* be a non-empty set and $f: H \times H \times H \to \wp^*(H)$, where $\wp^*(H)$ is the set of all non-empty subsets of *H*. Then f is called a *ternary hyperoperation* on *H* and the pair (H, f) is called a *ternary hypergroupoid*. If A, B, C are non-empty subsets of *H*, then we define

$$f(A,B,C) = \bigcup_{a \in A, b \in B, c \in C} f(a,b,c).$$

Definition 6. The ternary hypergroupoid (H, f) is called a *ternary semihypergroup* if for every $a_1, \ldots a_5 \in H$, we have

$$f(f(a_1,a_2,a_3),a_4,a_5) = f(a_1,f(a_2,a_3,a_4),a_5) = f(a_1,a_2,f(a_3,a_4,a_5)).$$

Definition 7. A ternary semihypergroup (H,f) is called a *ternary hypergroup* if for all $a,b,c \in H$ there exist $x,y,z \in H$ such that:

$$c \in f(x,a,b) \cap f(a,y,b) \cap f(a,b,z).$$

Notice that a ternary semigroup (S,f) is said to be a *ternary group* if it satisfies the following property that for all $a,b,c \in S$, there exist unique $x,y,z \in S$ such that

$$c = f(x,a,b),\ c = f(a,y,b),\ c = f(a,b,z).$$

Definition 8. A *ternary polygroup* is a multi-valued system $\langle P,f,e,^{-1}\rangle$ where $e \in P$, $^{-1}: P \to P$ is a unitary operation and * is a ternary hyperoperation from $P \times P \times P$ into the family of non-empty subsets of P such that the following axioms hold:

(1) $f(f(a_1,a_2,a_3),a_4,a_5) = f(a_1,f(a_2,a_3,a_4),a_5) = f(a_1,a_2,f(a_3,a_4,a_5))$, for every $a_1,\dots a_5 \in P$,

(2) is a unique element such that $f(x,e,e) = f(e,x,e) = f(e,e,x) = x$, for every $x \in P$, and $e^{-1} = e$,

(3) $z \in f(x_1,x_2,x_3)$ implies $x_1 \in f(z,x_2^{-1},x_3^{-1})$, $x_2 \in f(x_1^{-1},z,x_3^{-1})$ and $x_3 \in f(x_1^{-1},x_2^{-1},z)$.

Fuzzy Ternary Subpolygroups

Definition 9. Let $\langle P,f,e,^{-1}\rangle$ be a ternary polygroup and μ be a fuzzy subset of P. Then, μ is said to be a *fuzzy ternary subpolygroup* of P if the following axioms hold:

(1) $\min\{\mu(x),\mu(y),\mu(z)\} \le \inf\limits_{z \in f(x,y,z)} \{\mu(z)\}$ for all $x,y,z \in P$,

(2) $\mu(x) \le \mu(x^{-1})$ for all $x \in P$.

For any fuzzy subset μ of a non-empty set X and any $t \in (0,1]$, we define the set $U(\mu;t) = \{x \in X \mid \mu(x) \ge t\}$.

Theorem 1. *Let $\langle P,f,e,^{-1}\rangle$ be a ternary polygroup and μ be a fuzzy subset of P. Then, μ is a fuzzy ternary subpolygroup of P if and only if for every $t \in (0,1]$, $U(\mu;t)$ ($\neq \varnothing$) is a ternary subpolygroup of P.*

Proof. Suppose that μ is a fuzzy ternary subpolygroup of P. For every $x,y,z \in U(\mu;t)$ we have $\min\{\mu(x),\mu(y),\mu(z)\} \ge t$ and so $\inf_{a \in f(x,y,z)}\{\mu(a)\} \ge t$. Thus, for every $a \in f(x,y,z)$ we have $\mu(a) \ge t$. Therefore, $f(x,y,z) \subseteq U(\mu;t)$.

Now, if $x \in U(\mu;t)$ then $t \le \mu(x)$. Since $\mu(x) \le \mu(x^{-1})$, we conclude that $t \le \mu(x^{-1})$ which implies that $x^{-1} \in U(\mu;t)$.

Conversely, assume that for every $0 \le t \le 1$, $U(\mu;t)$ ($\neq \varnothing$) is a ternary subpolygroup of P. For every $x,y,z \in P$, we put $t_0 = \min\{\mu(x),\mu(y),\mu(z)\}$. Then $x,y,z \in U(\mu;t_0)$ and so $f(x,y,z) \subseteq U(\mu;t_0)$. Therefore, for every $a \in f(x,y,z)$ we have $\mu(a) \ge t_0$ implying that $\min\{\mu(x),\mu(y),\mu(z)\} \le \inf\limits_{a \in f(x,y,z)} \{\mu(a)\}$ and in this way the first condition of Definition 9 is verified. In order to verify the second condition, let $x \in P$. We put $t_1 = \mu(x)\}$. Since $U(\mu;t_1)$ is a ternary subpolygroup, $x^{-1} \in U(\mu;t_1)$, which implies that $\mu(x) \le \mu(x^{-1})$.

Anti Fuzzy Ternary Subpolygroups

Definition 10. Let $\langle P, f, e, ^{-1}\rangle$ be a ternary polygroup and μ be a fuzzy subset of *P*. Then, μ is said to be an *anti fuzzy ternary subplygroup P* if the following axioms hold:

(1) $\sup_{\alpha \in f(x,y,z)} \{\mu(z)\} \leq \max\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in P$,

(2) $\mu(x^{-1}) \leq \mu(x)$ for all $x \in P$.

For any fuzzy subset μ of a non-empty set X and any $t \in (0,1]$, we define the set $L(\mu;t) = \{x \in X \mid \mu(x) \leq t\}$.

Theorem 2. *Let* $\langle P, f, e, ^{-1}\rangle$ *be a ternary polygroup and* μ *be a fuzzy subset of P. Then,* μ *is an anti fuzzy ternary subpolygroup of P if and only if for every* $t \in (0,1]$, $L(\mu;t)$ $(\neq \varnothing)$ *is a ternary subpolygroup of P.*

Proof. Suppose that μ is an anti fuzzy ternary subpolygroup of *P*. For every $x, y, z \in L(\mu;t)$ we have $\max\{\mu(x), \mu(y), \mu(z)\} \leq t$ and so $\sup_{a \in f(x,y,z)}\{\mu(a)\} \leq t$. Thus, for every $a \in f(x,y,z)$ we have $\mu(a) \leq t$. Therefore, $f(x,y,z) \subseteq L(\mu;t)$.

Now, if $x \in L(\mu;t)$ then $x \in L(\mu;t)$ Since $\mu(x^{-1}) \leq \mu(x)$, we conclude that $\mu(x^{-1}) \leq t$ which implies that $x^{-1} \in L(\mu;t)$.

Conversely, assume that for every $0 \leq t \leq 1$, $L(\mu;t)$ $(\neq \varnothing)$ is a ternary subhypergroup of *H*. For every $x, y, z \in H$, we put $t_0 = \max\{\mu(x), \mu(y), \mu(z)\}$. Then $x, y, z \in L(\mu;t_0)$ and so $f(x,y,z) \subseteq L(\mu;t_0)$. Therefore, for every $a \in f(x,y,z)$ we have $\mu(a) \leq t_0$ implying that

$$\sup_{a \in f(x,y,z)} \{\mu(a)\} \leq \max\{\mu(x), \mu(y), \mu(z)\}$$

and in this way the first condition of Definition 10 is verified. In order to verify the second condition, let $x \in P$. We put $x \in P$. Since $L(\mu;t_1)$ is a ternary subpolygroup, $x^{-1} \in L(\mu;t_1)$, which implies that $\mu(x^{-1}) \leq \mu(x)$.

Theorem 3. *Let* $\langle P, f, e, ^{-1}\rangle$ *be a ternary polygroup and* μ *be a fuzzy subset of P. Then,* μ *is a fuzzy ternary polygroup of P if and only if it's complement* μ^c *is an anti fuzzy ternary hypergroup of P.*

Proof. Suppose that μ is a fuzzy ternary polygroup of P. For every x, y, z in *P*, we have

$$\min\{\mu(x), \mu(y), \mu(z)\} \leq \inf_{a \in f(x,y,z)}\{\mu(a)\}, \text{ or}$$

$$\min\{1-\mu^c(x), 1-\mu^c(y), 1-\mu^c(z)\} \leq \inf_{a \in f(x,y,z)}\{1-\mu^c(a)\}, \text{ or}$$

$$\min\{1-\mu^c(x), 1-\mu^c(y), \mu^c(z)\} \leq 1-\sup_{a \in f(x,y,z)}\{\mu^c(a)\}, \text{ or}$$

$$\sup_{a \in f(x,y,z)}\{\mu^c(\alpha)\} \leq 1-\min\{1-\mu^c(x), 1-\mu^c(y), 1-\mu^c(z)\}, \text{ or}$$

$$\sup_{a \in f(x,y,z)}\{\mu^c(\alpha)\} \leq \max\{\mu^c(x), \mu^c(y), \mu^c(z)\}.$$

And in this way the condition (1) of Definition 10 is verified for μ^c.

Since μ is a fuzzy ternary subpolygroup of *P*, so for every $x \in P$, $\mu(x) \le \mu(x^{-1})$ or $1-\mu(x^{-1}) \le 1-\mu(x)$ which implies that $\mu^c(x^{-1}) \le \mu^c(x)$ and the second condition of Definition 10 is satisfied. Therefore, μ^c is an anti ternary semipolygroup of *P*.

The converse also can be proved similarly.

Proposition 1. Let μ be an anti fuzzy ternary subpolygroup of a ternary polygroup $\langle P, f, e, ^{-1} \rangle$ and $t_1 < t_2$. Then $L(\mu,t_1)$ and $L(\mu,t_2)$ are equal if there is no $x \in P$ such that $t_1 < \mu(x) \le t_2$.

Proof. Suppose that $L(\mu,t_1) = L(\mu,t_2)$. If there exists $x \in P$ such that $t_1 < \mu(x) \le t_2$, then $L(\mu,t_1)$ is a proper subset of $L(\mu,t_2)$ and this is a contradiction.

Conversely, assume that there is no $x \in P$ such that $t_1 < \mu(x) \le t_2$. Since $t_1 < \mu(x) \le t_2$, $L(\mu,t_1) \subseteq L(\mu,t_2)$. Now, let $x \in L(\mu,t_2)$ be an arbitrary element. Then $\mu(x) \le t_2$ and so by hypothesis we conclude that $\mu(x) \le t_1$. Thus, $x \in L(\mu,t_1)$. Therefore, $L(\mu,t_1) = L(\mu,t_2)$.

Proposition 2. Let $\langle P, f, e, ^{-1} \rangle$ be a ternary plygroup and μ be an anti fuzzy ternary subpolygroup of *P*. If $\text{Im}\,(\mu) = \{t_0, t_1, \ldots, t_n\}$ where $t_0 < t_1 < \ldots < t_n$, then $L(\mu;t_i)$ (for $i = 1,\ldots,n$) constitute all the lower level subsets of μ.

Proof. Suppose that $t \in [0,1]$ and $t \notin Im(\mu)$. If $t > t_n$, then $L(\mu,t_n) \subseteq L(\mu,t)$. Since $L(\mu;t_n) = P$, it follows that $L(\mu;t) = P$ and so $L(\mu;t_n) = L(\mu;t)$. Now, suppose that $t_i < t < t_{i+1}$ where $1 \le i \le n-2$. Then there is no $x \in P$ such that $x \in P$. By Proposition 1, we have $L(\mu;t) = L(\mu;t_{i+1})$. This completes the proof.

Proposition 3. Let $\langle P, f, e, ^{-1} \rangle$ be a ternary polygroup and *K* be a non-empty subset of *P*. Let μ be a fuzzy subset of *P* such that

$$\mu(x) = \begin{cases} 0 & \text{if } x \in K \\ 1 & \text{if } x \notin K. \end{cases}$$

Then μ is an anti fuzzy ternary subpolygroup of *P* if and only if *K* is a ternary subpolygroup of *P*.

Proof. It is straightforward.

Theorem 4. *Let* $\langle P, f, e, ^{-1} \rangle$ *be a ternary hypergroup and K be a ternary subpolygroup of P. Then for each* $t \in [0,1]$, *there exists an anti fuzzy subpolygroup* μ *of P such that* $L(\mu;t) = K$.

Proof. Suppose that $t \in [0,1]$ and define a fuzzy subset μ of *P* by

$$\mu(x) = \begin{cases} t & \text{if } x \in K \\ 1 & \text{if } x \notin K. \end{cases}$$

Then, $L(\mu;t) = K$.

Let X_1, X_2 be two non-empty sets and μ, λ be fuzzy subsets of X_1, X_2, respectively. Then the product of μ and λ is the fuzzy subset $\mu \times \lambda$ of $X_1 \times X_2$ where

$$(\mu \times \lambda)(x,y) = \min\{\mu(x), \lambda(y)\}$$

for all $(x,y) \in X_1 \times X_2$.

Theorem 5. *Let* $\langle P_1, f_1, e_1, ^{-1} \rangle$ *and* $\langle P_2, f_2, e_2, ^{-1} \rangle$ *be two ternary polygroups and* μ, λ *be anti fuzzy* *ternary subpolygroups of* P_1, P_2, *respectively. Then* $\mu\times\lambda$ *is an anti fuzzy ternary subhypergroup of* $P_1\times P_2$.

Proof. Suppose that

$(x_1,x_2),(y_1,y_2),(z_1,z_2)\in P_1\times P_2$. For every $(a_1,a_2)\in(f_1\times f_2)((x_1,x_2),(y_1,y_2),(z_1,z_2))$ we have

$$\begin{aligned}(\mu\times\lambda)(a_1,a_2) &= \min\{\mu(a_1),\lambda(a_2)\}\\ &\le \min\{\max\{\mu(x_1),\mu(y_1),\mu(z_1)\},\max\{\lambda(x_2),\lambda(y_2),\lambda(z_2)\}\}\\ &= \min\{\max\{\mu(x_1),\lambda(x_2)\},\max\{\mu(y_1),\lambda(y_2)\},\max\{\mu(z_1),\lambda(z_2)\}\}\\ &= \max\{\min\{\mu(x_1),\lambda(x_2)\},\min\{\mu(y_1),\lambda(y_2)\},\min\{\mu(z_1),\lambda(z_2)\}\}\\ &= \max\{(\mu\times\lambda)(x_1,x_2),(\mu\times\lambda)(y_1,y_2),(\mu\times\lambda)(z_1,z_2)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_{(a_1,a_2)\in(f_1\times f_2)((x_1,x_2),(y_1,y_2),(z_1,z_2))}\{(\mu\times\lambda)(a_1,a_2)\}\\ &\le \max\{(\mu\times\lambda)(x_1,x_2),(\mu\times\lambda)(y_1,y_2),(\mu\times\lambda)(z_1,z_2)\}.\end{aligned}$$

So, the first condition of Definition 10 is satisfied. Similarly, we can prove the second condition of Definition 10.

Definition 11. Let $\langle P_1,f_1,e_1,^{-1}\rangle$ and $\langle P_2,f_2,e_2,^{-1}\rangle$ be two ternary polygroups and φ be a function from P_1 into P_2.

(1) If λ is a fuzzy subset of P_2, then the *preimage* of λ under φ is the fuzzy subset of P_1 defined by $\varphi^{-1}(\lambda)(x)=\lambda(\varphi(x))$, for all $x\in P_1$.

(2) If μ is a fuzzy subset of P_1, then the *image* of μ under φ is the fuzzy subset of P_2 defined by

$$\varphi(\mu)(y)=\sup_{x\in\varphi^{-1}(y)}\{\mu(x)\},\ \text{if } \varphi^{-1}(y)\neq\varnothing \text{ and } \varphi(\mu)(y)=0, \text{ otherwise.}$$

Definition 12. Let $\langle P_1,f_1,e_1,^{-1}\rangle$ and $\langle P_2,f_2,e_2,^{-1}\rangle$ be two ternary polygroups. A mapping φ from P_1 into P_2 is said to be a *strong homomorphism* if for every $x,y\in P_1$,

(1) $\varphi(e_1)=e_2$,

(2) $\varphi(f_1(x,y,z))=f_2(\varphi(x_1),\varphi(x_2),\varphi(x_3))$.

It is easy to see that $\varphi(x)^{-1}=\varphi(x^{-1})$.

Proposition 4. Let $\langle P_1,f_1,e_1,^{-1}\rangle$ and $\langle P_2,f_2,e_2,^{-1}\rangle$ be two ternary polygroups and φ be a strong homomorphism from P_1 onto P_2.

(1) If λ is a fuzzy ternary subpolygroup of P_2, then $\varphi^{-1}(\lambda)$ is a fuzzy ternary subpolygroup of P_1.

(2) If μ is a fuzzy ternary subpolygroup of P_1, then $\varphi(\mu)$ is a fuzzy ternary subpolygroup P_2.

Proof. It is straightforward.

Definition 13. Let $\langle P_1,f_1,e_1,^{-1}\rangle$ and $\langle P_2,f_2,e_2,^{-1}\rangle$ be two ternary polygroups and φ be a function from P_1 into P_2. If μ is a fuzzy subset of P_1, then the *anti image* of μ under φ is the fuzzy subset of P_2 defined by

$$\bar{\varphi}(\mu)(y) = \inf_{x \in \varphi^{-1}(y)} \{\mu(x)\}, \text{ if } \varphi^{-1}(y) \neq \varnothing \text{ and } \varphi(\mu)(y) = 1, \text{otherwise.}$$

Proposition 5. Let $\langle P_1, f_1, e_1, ^{-1} \rangle$ and $\langle P_2, f_2, e_2, ^{-1} \rangle$ be two ternary polygroups and φ be a strong homomorphism from P_1 onto P_2.

(1) If λ is a fuzzy ternary subpolygroup of P_2, then $\varphi^{-1}(\lambda^c) = (\varphi(\lambda))^c$.

(2) If μ is a fuzzy ternary subpolygroup of P_1, then $\varphi(\mu^c) = (\bar{\varphi}(\mu))^c$ and $\bar{\varphi}(\mu^c) = (\varphi(\mu))^c$.

Proof. (1) Suppose that λ is a fuzzy ternary subpolygroup of P_2. Then, for every $x \in P_1$,

$$\varphi^{-1}(\lambda^c)(x) = \lambda^c(\varphi(x)) = 1 - \lambda(\varphi(x)) = 1 - \varphi^{-1}(\lambda)(x) = (\varphi(\lambda))^c(x).$$

(2) Suppose that μ is a fuzzy ternary subpolygroup of P_1. Then, for every $y \in P_2$,

$$\varphi(\mu^c)(y) = \sup_{x \in \varphi^{-1}(y)} \left\{\mu^c(x)\right\} = \sup_{x \in \varphi^{-1}(y)} \{1 - \mu(x)\} = 1 - \inf_{x \in \varphi^{-1}(y)} \{\mu(x)\} = 1 - \bar{\varphi}(\mu)(y) = (\bar{\varphi}(\mu))^c(y).$$

Similarly, we obtain

$$\bar{\varphi}(\mu^c)(y) = \inf_{x \in \varphi^{-1}(y)} \left\{\mu^c(x)\right\} = \inf_{x \in \varphi^{-1}(y)} \{1 - \mu(x)\} = 1 - \sup_{x \in \varphi^{-1}(y)} \{\mu(x)\} = 1 - \varphi(\mu)(y) = (\varphi(\mu))^c(y).$$

Now, the proof is completed.

Theorem 6. *Let* $\langle P_1, f_1, e_1, ^{-1} \rangle$ *and* $\langle P_2, f_2, e_2, ^{-1} \rangle$ *be two ternary polygroups and* φ *be a strong homomorphism from* P_1 *onto* P_2.

(1) If λ *is an anti fuzzy ternary subpolygroup of* P_2, *then* $\varphi^{-1}(\lambda)$ *is an anti fuzzy ternary subpolygroup of* P_1.

(2) If μ *is an anti fuzzy ternary subpolygroup of* P_1, *then* $\bar{\varphi}(\mu)$ *is an anti fuzzy ternary subpolygroup* P_2.

Proof. (1) Suppose that λ is an anti fuzzy ternary subpolygroup of P_2. Then, by Theorem 3, λ^c is a fuzzy ternary subpolygroup of P_2. Hence, by Proposition 4, $\varphi^{-1}(\lambda^c)$ is a fuzzy ternary subpolygroup of P_1. Thus, by Proposition 5, $(\varphi^{-1}(\lambda))^c$ is a fuzzy ternary subpolygroup of P_1 and so, $\varphi^{-1}(\lambda)$ is an anti fuzzy ternary subpolygroup of P_1.

(2) Suppose that μ is an anti fuzzy ternary subpolygroup of P_1. Then, by Theorem 3, μ^c is a fuzzy ternary subpolygroup of P_1. Hence, by Proposition 4, P_1. is a fuzzy ternary subpolygroup of P_2. Now, by Proposition 5, $\varphi(\mu^c) = (\bar{\varphi}(\mu))^c$, so we conclude that $(\bar{\varphi}(\mu))^c$ is a fuzzy ternary subpolygroup of P_2. Therefore, $\bar{\varphi}(\mu)$ is an anti fuzzy ternary subpolygroup of P_2.

Conclusion

In the present paper, we defined the notion of anti fuzzy ternary subpolygroups of a ternary polygroup and obtained some related properties. In particular, we provided the relation between an anti fuzzy ternary subpolygroups and level ternary subpolygroups. This relation is expressed in terms of a necessary and sufficient condition. Our further research will consider the characterizations of ternary subpolygroups by using lattices.

მათემატიკა

ქვედა დონის სიმრავლეები და ანტიფუზი ტერნარული პოლიჯგუფები

ბ. დავაზი

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(წარმოდგენილია აკადემიკოს ვ. კოკილაშვილის მიერ)

ტერნარული პოლიჯგუფის ცნება განაზოგადებს კომერის აზრით პოლიჯგუფის ცნებას. ნაშრომში შესწავლილია ტერნარული პოლიჯგუფის ფუზი და ანტიფუზი ტერნარული ქვეჯგუფები და დადგენილია მათი თვისებები.

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