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Mathematics

Primitive Elements of Free Lie *p*-Algebras

Giorgi Rakviashvili

Faculty of Science and Arts, Ilia State University, Tbilisi

(Presented by Academy Member Hvedri Inassaridze)

ABSTRACT. Let *L* be a finitely generated free Lie *p*-algebra and $\langle a \rangle$ an ideal generated by $a \in L$. It is proved that $L/\langle a \rangle$ is free if and only if *a* is primitive (i.e. *a* belongs to some set of free generators of *F*). Earlier analogous theorems were proved for some objects, for example for groups, Lie algebras, free algebras and etc. *© 2014 Bull. Georg. Natl. Acad. Sci.*

Key words: *free Lie p-algebra, primitive element, Freiheitssatz.*

It is known [1] that if *F* is a finitely generated free group and $a \in F$, then *a* is a primitive element (i.e. *a* belongs to some set of free generators of *F)* if and only if $F/\langle a \rangle$ is a free group ($\langle a \rangle$ denotes a normal subgroup of *F* generated with *a*). Later similar theorems were proved for Lie algebras [2], free algebras, free commutative algebras and free anticommutative algebras [3]. Mikhalev, Shpilrain and Umirbaev in [4] conjectured that analogous theorem for Lie *p*-algebras is also true. In [5] the author proved Freiheitssatz for Lie p-algebras but it seems impossible to prove foresaid theorem with it. In this paper we prove a theorem about primitive elements of free Lie p-algebras in the same manner as in [2] using Bokut's result from [6].

Let k be a field of the characteristic $p > 0,\ p \neq 2$, let $X = \{x_1, x_2, ..., x_n\}$ be a finite set and $F = k\langle X \rangle$ a free associative algebra without identity with X as a set of free generators. We assume that $x_i < x_j \Leftrightarrow i > j$ and if w_1 and w_2 are words from $k\langle X \rangle$, then $w_1 < w_2$ either $\deg w_1 < \deg w_2$ or $\deg w_1 = \deg w_2$ and $w_1 < w_2$ lexicographically.

If $f \in F = k\langle X \rangle$ let $\overline{f}$ denotes a leading word of *F* with nonzero coefficients (particularly coefficients of $\overline{f}$ is equal to one). It is clear that $\overline{fg} = \overline{f}\,\overline{g}$.

Let $L_p\langle X \rangle$ denotes a free Lie *p*-algebra over *k* by *X* as a set of free generators. A set $Y \subset L_p\langle X \rangle$ is called *p-independent* [2] if *Y* is the set of free generators of Lie *p*-subalgebra of $L_p\langle X \rangle$ generated by *Y* (recall that any Lie *p*-subalgebra of free Lie *p*-algebra is free [7]).

Now we shall recall several definitions and results about $L_p\langle X\rangle$.

A linear basis of $L_p\langle X\rangle$ are *p*-proper words [2] which are formed from symbols $\{x_1, x_2, ..., x_n\}$. If $L\langle X\rangle$ denotes a free Lie algebra free generated from the set *X*, then the proper words of $L_p\langle X\rangle$ are formal p^k-degrees of proper words of $L\langle X\rangle$.

We shall use ordinary concept of degree of elements from $L_p\langle X\rangle$, for example, if $f = x_\alpha x_\beta + x_\gamma^{p^2}$ then $\deg f = p^2$. We assume that deg 0=0.

Suppose $f \in L_p\langle X\rangle$, $f = \sum_i \alpha_i q_i$, where q_i are *p*-proper words. Such a record of *f* is called a *right form* of *f*. An element $f' = \sum_{i\in I} \alpha_i q_i$ where $\deg q_i = \deg f$ and $\deg q_i < \deg f$ if $i \notin I$ is called a *major part* of *f*. Let $\tilde{f}$ denotes *major member* of $f \in L_p\langle X\rangle$ - a lexicographically major word among q_i, $i \in I$ (see [2]).

A subset $Y \subset L_p\langle X\rangle$ is called *p-reduced* [8], if for any $f \in Y$ its major part f' does not belong to Lie *p*-subalgebra of $L_p\langle X\rangle$, which is generated by major parts of all elements from $Y\setminus\{f\}$. We assume that empty set is *p*-reduced.

Let $Y = \{y_i\} \subset L_p\langle X\rangle$ be a finite set. A map $t: Y \to L_p\langle X\rangle$ is called elementary, if for some *j*

$$t(y_i) = y_i, \text{ if } i \neq j,$$

$$t(y_i) = \alpha y_j + \varphi(y_{i_1}, y_{i_2}, ..., y_{i_m}), \text{ if } i_1, i_2, ..., i_m \neq j,$$

where $\alpha \in k, \alpha \neq 0$ and φ is a polynomial, i.e. an element of free Lie *p*-algebra with *m* free generators.

Suppose $\{y_i\}$ is finite. Let Y' denote a set of major parts of elements from *Y* regarding to standard ordering considered in the beginning of the paper. Let also

$$l(Y) = \sum_i y_i,$$

as we have noted $\deg y_i$ is the length of the longest word in y_i and deg 0=0.

Lemma 1. *Let* $\{y_1, y_2, ..., y_m\}$ *be a finite set of generators of* $L_p\langle X\rangle$. *Then there exist* (*l*(*Y*)-*n*) *elementary maps which translate Y onto a set of generators of* $L_p\langle X\rangle$ *with degrees* (*regarding to X*) *less or equal to one, where n is a number of free generators of* $L_p\langle X\rangle$)

Remark 1. This Lemma was proved in [2] for Lie algebras. We prove our Lemma in the same manner.

For $a \in k\langle X\rangle$ let $\langle a\rangle, a \in \langle a\rangle$, an ideal of $k\langle a\rangle$ generated by *a*.

Lemma 2. *If* $a, b \in k\langle X\rangle$ *and* $\langle a\rangle = \langle b\rangle$ *then a and b are linearly dependent.*

Proof. If either *a* or *b* is zero, our proposition of course is valid. So, we may assume that $a, b \neq 0$; then $\bar{a}, \bar{b} \neq 0$. From [6] it follows that if $x \in \langle a\rangle$ then $\bar{a}$ is a subword of $\bar{x}$. Therefore $\bar{a}$ is the subword of $\bar{b}$ and vice versa $\bar{b}$ is the subword of $\bar{a}$ and consequently $\bar{a} = \bar{b}$. Suppose

$$a = \alpha\bar{a} + ..., \; b = \beta\bar{b}, \; \alpha, \beta \in k, \; \alpha, \beta \neq 0.$$

Consider an element $c = a - \frac{\alpha}{\beta} b \in \langle a \rangle = \langle b \rangle$; if $c \neq 0$, then $\overline{c}$ is less than $\overline{a}$; on the other hand, $\overline{a}$ is the subword of $\overline{c}$ - contradiction; so $c = 0$.

Corollary 1. *Let* $F_1 = k\langle X \rangle_1$ *be a free associative algebra with identity, which is freely generated by X; suppose* $a, b \in F_1$ *and* $\langle a \rangle = \langle b \rangle$. *Then a and b are linearly dependent.*

Let $\langle a \rangle$ denote an ideal of $L_p\langle X \rangle$ generated by *a* (we assume $a \in \langle a \rangle$) and let $\overline{a}$ be a major word of *a*.

Corollary 2. *Let* $\langle a \rangle = \langle b \rangle \subset L_p\langle X \rangle$ *; then a and b are linearly dependent.*

An element $a \in L_p\langle X \rangle$ is primitive, if there exists a set *Y* of free generators of $L_p\langle X \rangle$ such that $a \in Y$.

Theorem. $L_p\langle X \rangle / \langle a \rangle$ *is free if and only if a is primitive.*

Proof. It is clear that if *a* is primitive then $L_p\langle X \rangle / \langle a \rangle$ is free. Suppose $L_p\langle X \rangle / \langle a \rangle$ is free and let us prove that *a* is primitive.

Let us denote $\overline{L} = L_p\langle X \rangle / \langle a \rangle$. It is clear that $\dim(\overline{L}/\overline{L}^2) \geq n-1$.

On the other hand, $L_p\langle X \rangle$ is a generalized nilpotent, i.e. the intersection of all its degrees is zero. Due to [9] all generalized nilpotent algebras are Hopf type, i.e. they are not isomorphic to their proper factor-algebras; consequently

$$rank\overline{L} = rank\overline{L}\left(L_p\langle X \rangle / \langle a \rangle\right) \leq n-1 .$$

But if $rank\overline{L} < n-1$ then $\dim \overline{L}/\overline{L}^2 < n-1$; so $rank\overline{L} = n-1$ and in $\overline{L}$ there exists a set of free generators $Y = \{y_1, y_2, ..., y_{n-1}\}$. The set $\overline{X} = \{\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\}$ generates $\overline{L}$ and due to Lemma 1 there exist elementary maps which translate $\overline{X}$ in such a set of generators $Z = \{z_1, z_2, ..., z_r\}$ of $\overline{L}$ that the degrees of each z_i with respect to *Y* are not greater than one. Also, there exist elementary maps which translate $Z = \{z_1, z_2, ..., z_r\}$ on $\{z_1, z_2, ..., z_{r_0}, 0, ..., 0\}$ where $\{z_1, z_2, ..., z_{r_0}\}$ is maximal linearly independent set in *Z*. It is clear that $r_0 = n-1$ and the number of zeros in $\{z_1, z_2, ..., z_{r_0}, 0, ..., 0\}$ is one, consequently, some elementary maps transform $\{z_1, z_2, ..., z_{r_0}, 0\}$ on $\{y_1, y_2, ..., y_{r_0}, 0\}$ (this set contains only zero, then $n = 1$). Therefore, we may assume that there exist the elementary maps $\varphi_1, \varphi_2, ..., \varphi_s$ which transform $\overline{X} = \{\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\}$ on $\{y_1, y_2, ..., y_{r_0}, 0\}$. The elements $X = \{x_1, x_2, ..., x_n\}$ are pre-images of $\overline{X} = \{\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\}$. The maps $\varphi_1, \varphi_2, ..., \varphi_s$ transform *X* on a set $\{t_1, t_2, ..., t_n\}$ of free generators of $L_p\langle X \rangle$. Let us consider a projection $\pi : L_p\langle X \rangle \to L_p\langle X \rangle / \langle a \rangle$. It is clear that $\pi(t_n) = 0$. So, $t_n \in \langle a \rangle$ i.e. $\langle t_n \rangle \subset \langle a \rangle$. Really $\langle t_n \rangle = \langle a \rangle$. Indeed, let us consider a free algebra $L_p\langle X \rangle / \langle t_n \rangle$. As $\langle t_n \rangle \subset \langle a \rangle$ so

$$(L_p\langle X \rangle / \langle t_n \rangle) / (\langle a \rangle / \langle t_n \rangle) \cong L_p\langle X \rangle / \langle a \rangle .$$

As $L_p\langle X \rangle / \langle t_n \rangle$ and $L_p\langle X \rangle / \langle a \rangle$ are free Lie *p*-algebras with $n-1$ generators and free Lie *p*-algebras are Hopf type algebras we must have $\langle a \rangle / \langle t_n \rangle = 0$ i.e. $\langle a \rangle = \langle t_n \rangle$. Then from Corollary 2 it follows that

$a = \alpha t_n$ for some , $\alpha \in k$ i.e. a is primitive.

Remark 2. We assume that in this way it is possible to prove other results from [2].

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თავისუფალი ლის p-ალგებრის პრიმიტიული ელემენტები

გ. რაქვიაშვილი

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