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Mathematics

Construction of Monadic Heyting Algebra in any Logos

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ABSTRACT. Heyting algebras have been investigated by Arend Heyting as semantics for intuitionistic Logic. The connection between heyting algebras and some types of categories(logos, topos) was noticed A. Grothendieck . Hence it was noticed that the extension of intuitionistic logic with modal connectives can be modeled by Heything algebras with additional operators. Connection of such extension with topological models for Heyting algebra has been investigated by L. Esakia and H. Ono. Heyting algebra with two operators, which was a model for a particular modal logic was constructed by H. Ono. In this paper the attempt was made to construct Heyting algebra with two operators in special categories, called logos, and to show the connection of this construction with H. Ono modal logic. *© 2014 Bull. Georg. Natl. Acad. Sci.*

Key words: *Heyting Algebra, logos, category theory, modal logics.*

Heyting algebra H (also called pseudo-Boolean algebra) is a poset with all finite products and coproducts, which is Cartesian Closed (as a category with products). In other words, Heyting algebra is a lattice with 0 and 1 which has for each pair of elements x, y an exponential y^x. This exponential is usually written as $x \Rightarrow y$. By its definition it is characterized by the adjunction:

$$z \le (x \Rightarrow y) \quad \text{iff} \quad z \wedge x \le y .$$

In other words, $x \Rightarrow y$ is the least upper bound for all those elements: with $z \wedge x \le y$ in particular, then $y \le (x \Rightarrow y)$. Thus, in the usual picture of a partially order $x \Rightarrow y$ lies above y.

In Boolean algebra for all x, y and z

$$z \le (\neg x \vee y) \quad \text{iff} \quad z \wedge x \le y .$$

Proof. Only if

$$z \wedge x \le (\neg x \vee y) \wedge x \le g \wedge x \le y .$$

If

$$z = z \wedge 1 = z \wedge (\neg x \vee x) = (z \wedge \neg x) \vee (z \wedge x) \le \neg x \vee y .$$

Hence Boolean algebra has exponentials given by $(x \Rightarrow y) = \neg x \vee y$. Therefore every Boolean algebra is Heyting algebra. The converse does not hold, for example, the open sets in the real line form Heyting algebra, which is not Boolean.For any topological space *X*, the set open (*X*) of all open sets in *X* is Heyting algebra. It is a lattice (under inclusion) because binary unions and intersections of open sets are open, as are the sets $\varnothing$ and *X*. For two open sets *U* and *V* the exponential $U \Rightarrow V$ can be defined as the union UW_i of all those open sets W_i for which $W_i \cap U \subset V$. Then because intersection is distributive over arbitrary unions:

$$(\cup W_i) \cap U = \cup (W_i \cap U) \subset V.$$

Therefore $\cup W_i = (U \Rightarrow V)$.

A similar argument will show that any complete and (infinitely) distributive lattice is Heyting algebra. (Here, a lattice is said to be complete when, regarded as a category, it has all small limits and small colimits, i.e. all small products and coproducts.)

Next we will introduce intuitionistic modal logic M_H (*H*-intuitionistic logic). The intuitionistic modal logic M_H is obtained from the intuitionistic propositional logic H_0 by adding the following axioms:

1) $\Box p \Rightarrow p$ 1′) $p \Rightarrow \Diamond p$

2) $\Box p \Rightarrow \Box\Box p$ 2′) $\Diamond\Diamond p \Rightarrow \Diamond p$

3) $\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$

4) $\Box(p \Rightarrow q) \Rightarrow (\Diamond p \Rightarrow \Diamond q)$

5) $\Diamond(p \vee q) \Rightarrow (\Diamond p \vee \Diamond q)$

6) $\Diamond p \Rightarrow \Box\Diamond p$ 6′) $\Diamond\Box p \Rightarrow \Box p$

7) $(\Box p \Rightarrow \Box q) \Rightarrow \Box(\Box p \Rightarrow \Box q)$

Rules of inference of M_H are modus ponens, the rule of substitution and the rule of necessitation, i.e. from *A* infer $\Box A$.

Let us take a one-to-one correspondence between propositional variables of M_H and monadic pre-variables. For each proposition variable *p* of M_H, $p^*(x)$ denotes the monadic, predicate variable corresponding to *p*. Also, we fix an individual variable *x*. Now ψ is defined as follows:

1) $\psi(p) = p^*(x)$ if *p* is a propositional variable

2) $\psi(A \wedge B) = \psi(A) \wedge \psi(B)$

3) $\psi(A \vee B) = \psi(A) \vee \psi(B)$

4) $\psi(A \Rightarrow B) = \psi(A) \Rightarrow \psi(B)$

5) $\psi(\neg A) = \neg\psi(A)$

6) $\psi(\Box A) = \forall x \psi(A)$

7) $\psi(\Diamond A) = \exists x \psi(A)$

Theorem 1 ([2]). *A formula A is provable in* M_H *if and only if* $\psi(A)$ *is provable in H* (*int. log.*).

Now we will introduce algebraic semantic for modal logic.

Definition ([1]). An algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, I, C, 0, 1 \rangle$ is said to be a bi-topological pseudo-Boolean (Heyting) algebra (bi-tp Ba) if

(1) $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is pseudo-Boolean (Heyting) algebra

(2) for each x and $y \in A$

(i)	$I(x \cap y) = Ix \cap Iy$	(i')	$C(x \cup y) = Cx \cup Cy$
(ii)	$Ix \leq x$	(ii')	$x \leq Cx$
(iii)	$IIx = Ix$	(iii')	$Cx = CCx$
(iv)	$I1 = 1$	(iv')	$C0 = 0$
(v)	$Ix \rightarrow Iy$	(v')	$I(x \rightarrow y) \leq Cx \rightarrow Cy$
(vi)	$CIx = Ix$	(vi')	$ICx = Ix$.

An assignment of bi-tpBa A is defined in the usual way. In particular, for each assignment f of A, $f(\Box A) = If(A)$ and $f(\Diamond A) = Cf(A)$. A modal formula A is said to be valid in a bi-tpBa, if $f(A) = 1$ for every assignment f of A. The set of all modal formulas valid in a bi-tpBa A is denoted by $L(A)$. Clearly, $L(A)$ is closed under modus ponens and the rule of necessitation.

Fact (1) ([1]).

1) For each bi-tpBa A $L(A) \supset M_H$.

2) For each formula ϕ such that $M_H \Rightarrow \phi$ there exists a bi-tpBa A such that $\phi \notin L(A)$.

Construction of Heyting Algebra with Two Operators

Now we will try to construct (for any b and X) from Heyting algebra $\mathrm{Sub}_b(X)$ abi-tpBa first define a monadic Heyting algebra $\langle A, \vee, \wedge, \rightarrow, I, C, 0, 1 \rangle$ with all properties similar to bi-tpBa exept (v') instead we have :

$$(\mathrm{v}'')\ C(p \wedge Cq) = Cp \wedge Cq .$$

Proposition. *In any monadic Heyting algebra* $p \leq q$ *implies* $Cp \leq Cq$.

Lemma. *Any monadic Heyting algebra is a bi-tpBa.*

Proof. After definition of $\rightarrow$ we have

$$r \leq p \rightarrow q \ \text{ iff } \ p \wedge r \leq q . \quad (1)$$

From $Ip \leq p$ ((ii)) we have $I(p \rightarrow q) \leq p \rightarrow q$ so from (1) $p \wedge I(p \rightarrow q) \leq q$ using Proposition II we get $C(p \wedge I(p \rightarrow q)) \leq Cq$. Property (vi) $(CIx = Ix)$ gives us $C(p \wedge CI(p \rightarrow q)) \leq Cq$ using (v'') and (iii) we get $Cp \wedge CI(p \rightarrow q) \leq Cq$ and again use (vi) $Cp \wedge I(p \rightarrow q)$, so applying (1) we have $I(p \rightarrow q) \leq Cq \rightarrow Cq$ and proof is finished. Now define in any logos C for any object X a construction $\langle Sub_C(X), \vee, \wedge, \rightarrow, I, C, \bot_x, \mathrm{T}_x \rangle$, where we take I, C to be $f^{-1} \circ \forall f$, $f^{-1} \circ \exists f$ ($f : X \rightarrow 1$ map from X to terminal object).

Theorem 2. $\langle Sub_C(X), \vee, \wedge, \rightarrow, I, C, \bot_x, \mathrm{T}_x \rangle$ *is a bi-tpBa for any X.*

Proof. Let us write the adjointness properties in the following form:

$$\frac{p \le f^{-1}(q)}{\exists f(p) \le q} \tag{1}$$

$$\frac{p \le \forall f(q)}{f^{-1}(p) \le q}. \tag{1'}$$

Now proove properties (ii), (ii') of bi-tpBa

(ii') $p \le Cp \Leftrightarrow p \le f^{-1}\exists f(p)$ ($\Leftrightarrow$ means equals) $\Leftrightarrow \exists f(p) \le \exists f(p)$ (used (1));

$$(\mathrm{ii})\ Ip \le p \Leftrightarrow f^{-1}\forall fp \le p \Leftrightarrow \forall f(p) \le \forall f(p) \tag{1'}$$

Proof (iii) and (iii')

(iii') $CCp = Cp \Leftrightarrow CCp = Cp$ and $CCp \ge Cp$

$CCp \ge Cp$ follows from (ii')

$CCp \le Cp \Leftrightarrow f^{-1}\exists ff^{-1}\exists fp \le f^{-1}\exists fp \Leftarrow$ ($\Leftarrow$ means follows from) $\Leftarrow$

$\Leftarrow \exists ff^{-1}\exists f(p) \le \exists f(p)$ (these we use the fact that f^{-1} is functor between posets) $\Leftrightarrow$

$$\Leftrightarrow f^{-1}\exists f(p) \le f^{-1}\exists f(p). \tag{1}$$

(iii) is similar.

Proof (i') and (i).

(i') $C(p \vee q) = Cp \vee Cq \Leftrightarrow C(p \vee q) \le Cp \vee Cq$ and $C(p \vee q) \ge Cp \vee Cq$.

$C(p \vee q) \ge Cp \vee Cq \Leftrightarrow f^{-1}\exists f(p \vee q) \ge f^{-1}\exists fp \vee f^{-1}\exists fq$ because $p \vee q \ge p$ and $p \vee q \ge q$, so using the fact that f^{-1} and $\exists f$ are functors we have $f^{-1}\exists f(p \vee q) \ge f^{-1}\exists fp \vee \exists fq$ (by definition of $\vee$)

$$f^{-1}\exists f(p \vee q) \le f^{-1}\exists fp \vee f^{-1}\exists fq \Leftarrow \exists f(p \vee q) \le \exists f(p) \vee \exists f(q) \Leftrightarrow$$
$$\Leftrightarrow (by\ \ (1)) \Leftrightarrow p \vee q \le f^{-1}(\exists f(p) \vee \exists f(q)) \Leftrightarrow p \vee q \le f^{-1}\exists f(p) \vee f^{-1}\exists f(q) \Leftrightarrow$$
$$\Leftrightarrow p \vee q \le Cp \vee Cq,$$

but we have from (ii') $p \le Cp$ and $q \le Cq$ so $p \vee q \le Cp \vee Cq$.

(i) Similar in proof of (i') we used the fact that f^{-1} preserves $\vee$ this follows from more general proposition that f^{-1} is Heyting algebra homomorphism.

Proof (iv') and (iv).

(iv') $C \perp_x = \perp_x \Leftrightarrow f^{-1}\exists f \perp_x = \perp_x \Leftrightarrow f^{-1}\exists f \perp_x \le \perp_x$ and $f^{-1}\exists f \perp_x \ge \perp_x$. $f^{-1}\exists f \perp_x \ge \perp_x$ is clear,

$f^{-1}\exists f \perp_x \le \perp_x \Leftrightarrow f^{-1}\exists f \perp_x \ge f^{-1}(\perp_1)$ (because f^{-1} is Heyting homomorphism) $\Leftrightarrow$

$\Leftrightarrow \exists f \perp_x \le \perp_1$ (by (1)) $\perp_x \le f^{-1}(\perp_1) \le \perp_x$

q.e.d.

(iv) is similar.

Let us prove now (v) and (v').

$$(\mathrm{v})\ Ip \to Iq \le I(Ip \to Iq) \Leftrightarrow f^{-1}\forall fp \to f^{-1}\forall fq \le f^{-1}\forall f\left(f^{-1}\forall fp \to f^{-1}\forall fq\right) \Leftrightarrow$$

$$\Leftrightarrow f^{-1}(\forall fp \to \forall fq) \le f^{-1}\forall f\left(f^{-1}\forall fp \to f^{-1}\forall fq\right) \Leftrightarrow$$
$$\Leftrightarrow \forall f^{-1}\forall fp \to \forall fq \le \forall f\left(f^{-1}\forall fp \to f^{-1}\forall fq\right)\ (\mathrm{by}\ (1)) \Leftrightarrow$$
$$\Leftrightarrow f^{-1}(\forall fp \to \forall fq) \le f^{-1}\forall fp \to f^{-1}\forall fq \Leftrightarrow$$
$$\Leftrightarrow f^{-1}\forall fp \to f^{-1}\forall fq \le f^{-1}\forall fp \to f^{-1}\forall fq .$$

(v') Instead of (v') we prove (v'') from which (v') follows:

$$C(p \wedge Cq) = Cp \wedge Cq \Leftrightarrow f^{-1}\exists f\left(p \wedge f^{-1}\exists fq\right) = f^{-1}\exists fp \wedge f^{-1}\exists fq \Leftarrow$$
$$\Leftarrow \exists f\left(p \wedge f^{-1}\exists fq\right) = \exists fp \wedge \exists fq,$$

which follows from Proposition I, if we take $I = p$ and $\exists fq = \exists fp$.

$(\mathrm{vi})\ CIp = Ip \Leftrightarrow CIp \ge Ip$ and $CIp \le Ip$ so $CIp \ge Ip$ is clear (by (ii')).

$$CIp \le Ip \Leftrightarrow f^{-1}\forall ff^{-1}\forall fp \le f^{-1}\forall fp \Leftarrow$$
$$\Leftarrow \exists ff^{-1}\forall fp \Leftarrow \exists ff^{-1}\forall fp \le \forall fp \Leftrightarrow\ (\mathrm{by}\ (1)) \Leftrightarrow$$
$$\Leftrightarrow f^{-1}\forall fp \le f^{-1}\forall fp.$$

The proof of (vi') is similar, and so the whole proof of theorem is finished.

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(წარმოდგენილია აკადემიკოს ხ. ინასარიძის მიერ)

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