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Mathematics

On One Singular Integral Equation Arising from the Radiative Transfer Theory

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ABSTRACT. In the class of Hölder functions we give the necessary and sufficient condition for solvability of the two-dimensional integral equation having singularity with respect to one variable. Such equations often arise from the radiative transfer theory. Finding a solution is reduced to solving a one dimensional Fredholm integral equation of the second kind. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *singular integral, piecewise function, radiative transfer.*

We present a method for solving singular integral equation occuring while investigating many important problems of neutron transport theory [1,2]. This equation has the form

$$(Lu)(x,t) \equiv \int_a^b \int_{-1}^{+1} \frac{sk(x,y)}{s-t} u(y,s)dsdy + u(x,t) + \int_a^b \int_{-1}^{+1} \frac{tk(x,y)}{s-t} u(y,t)dsdy = f(x,t), \quad (1)$$

$$x \in [a,b],\ t \in (-1,+1)$$

where $k(x,y)$ is the real valued continuous function, the right part $f(x,t)$ is the real valued function satisfying H^* condition [3] with respect to *t*. Further, we shall denote the class of such functions by D^*. We look for solutions $u(x,t) \in D^*$. To our opinion it appears to be more natural and appropriate in certain applications of equations. Note that in this paper we generalize the result from [6], therefore formalism sometimes used here is similar to the formalism, which is used in [6].

Let us introduce the linear integral operator defined by formula:

$$(\Omega_z g_z)(x,t) \equiv g_z(x,t) + z \int_a^b \int_{-1}^{+1} \frac{k(x,y)}{s-z} g_z(y,s)dsdy,$$

$$z \notin [-1,+1],\ \ x \in [a,b],\ t \in (-1,+1),$$

where the parameter z is any point on the plane. The boundary properties of the introduced operator are of great importance for further investigation of the operator L. The operator Ω_z operating on any continuous function $g_z(x,t)$ piecewise holomorphic in z with a cut on the real axis $[-1,+1]$ and satisfying the H^* condition in t will define a piecewise holomorphic function with a cut on $[-1,+1]$. Using the Plemelj-Sokhotski formulas [3], we can calculate the limiting values of $g_z(x,t)$ as

$$\Omega_{\varsigma t}^{\pm} g_{\varsigma}^{\pm}(x,t) = g_{\varsigma}^{\pm}(x,t) + \varsigma \int_a^b \int_{-1}^{+1} \frac{k(x,y)}{s-\varsigma} g_{\varsigma}^{\pm}(y,t) ds dy \pm i\pi \int_a^b k(x,y) g_{\varsigma}^{\pm}(y,t) dy, \qquad (2)$$

$$\varsigma \in (-1,+1), \ x \in [a,b], \ t \in (-1,+1).$$

Denote by $\Re$ the set of values of z, for which the homogeneous equation

$$\Omega_z u = 0 \qquad (3)$$

admits non-zero solution. Such values of z are called eigenvalues of Ω_z. In as much as the kernel of the operator Ω is a piecewise analytic function in z, satisfying the H condition [3] and bounded as $z \to \infty$, from Tamarkin's Theorem [4] it follows that the set $\Re$ is no more than countable in the plane with a cut on $[-1,+1]$.

It is not difficult to prove that:

I_1. If $g_{z_k}(x,t)$ is a solution of Eq.(3), then the function

$$\tau_{z_k}(x,t) = \frac{z_k g_{z_k}(x,t)}{z_k - t},$$

where $z_k \in \Re$ is a solution of the equation

$$(z-t)\tau_z(x,t) - z \int_a^b \int_{-1}^{+} k(x,y) \tau_z(y,s) ds dy = 0,$$

$$x \in [a,b], \ t \in (-1,+1),$$

when $z = z_k$ and vice versa. In addition $\Re \cap [-1,+1] = \varnothing$.

I_2. If $z = z_k$ is an eigenvalue of multiplicity r_0 for the operator Ω_z, then z_k also is an eigenvalue of multiplicity r_0 for the operator

$$(\Omega_z^* q_z)(x,t) \equiv q_z(x,t) + z \int_a^b \int_{-1}^{+1} \frac{k(y,x)}{s-z} q_z(y,s) ds dy$$

$$z \notin [-1,+1], \ x \in [a,b], \ t \in (-1,+1),$$

and vice versa.

I_3. If $z \neq z'$, then

$$\int_a^b \int_{-1}^{+1} t \tau_z^*(x,t) \tau_{z'}(x,t) dt dx = 0,$$

where $\tau_z^*(x,t)$ is a solution of the equation

$$(z-t)\tau_z^*(x,t) - z\int_a^b\int_{-1}^{+1} k(y,x)\tau_z^*(y,s)dsdy = 0,$$

$$x\in[a,b],\ t\in(-1,+1).$$

In the sequel we consider the case when $\Re$ is the finite set. The functions $\tau_{z_k}^*(x,t)$ and $\tau_{z_k}(x,t)$ $z_k\in\Re$, are called eigenfunctions of the kernel $k(y,x)$ and $k(x,y)$, respectively. The basic result for L is the following theorem.

Let $f(x,t)\in D^*$.

Theorem. *For the singular integral equation*

$$Lu = f \tag{4}$$

to have a solution in the class D^*, it is necessary and sufficient that the function f satisfy the conditions

$$\int_a^b\int_{-1}^{+1} t\tau_{z_k}^*(x,t)f(x,t)dtdx = 0\,,\quad z_k\in\Re\,. \tag{5}$$

Proof. Proof of the necessity. Let us introduce into consideration the piecewise holomorphic function

$$\psi_z(x) = \frac{1}{2\pi i}\int_a^b\int_{-1}^{+1}\frac{sk(x,y)}{s-z}u(y,s)dsdy\,,$$

where z is an arbitrary point on the plane. This function possesses the following properties:

$(P)_1$. In the plane with the cut $[-1,+1]$, it is piecewise holomorphic with respect to z.

$(P)_2$. As $z\to\infty$ it vanishes uniformly in x.

$(P)_3$. By using the Plemelj-Sokhotski formulas

$$\psi_t^{\pm}(x) = \frac{1}{2\pi i}\int_a^b\int_{-1}^{+1}\frac{sk(x,y)}{s-t}u(y,s)dsdy \pm \frac{t}{2}\int_a^b k(x,y)u(y,t)dy$$

$$x\in[a,b],\ t\in(-1,+1).$$

Combining the above equalities with (2), in view of (1) we obtain that the function $\psi_z(x)$ will be solution of the following boundary value problem:

$$(\Omega_t^+\psi_t^+)(x) - (\Omega_t^-\psi_t^-)(x) = t\int_a^b k(x,y)f(y,t)dy\,,$$

$$x\in[a,b],\ t\in(-1,+1).$$

Consequently, taking the Plemelj-Sokhotski formulas into account, we can write

$$(\Omega_z\psi_z)(x) = \frac{1}{2\pi i}\int_a^b\int_{-1}^{+1}\frac{sk(x,y)}{s-z}f(y,s)dsdy\,,$$

$$x\in[a,b],\ t\in(-1,+1).$$

For this integral equation to define the analytic function $\psi_z(x)$ in a plane with a cut $[-1,+1]$, it is necessary and sufficient that its right-hand side satisfy the conditions

$$\int_a^b \int_{-1}^{+1} \tau_{z_k}^*(x,t) \int_a^b \int_{-1}^{+1} \frac{sk(x,y)}{s-z_k} f(y,s)dsdydtdx = 0 .$$

After simplification we get equality (5). The necessity is proved.

Proof of the Sufficiency. Let the function $f(x,t)$ satisfy the conditions (5).

By virtue of Tamarkin's theorem [5] it follows that this solution possesses the properties:

(R_1). It is piecewise holomorphic with respect to *z*.

(R_2). As $z \to \infty$ it vanishes uniformly in *x*.

(R_3). It may be represented in the form:

$$\psi_z(x) = \frac{1}{2\pi i} \int_{-1}^{+1} \frac{sm_s(x)}{s-z} ds ,$$

$$z \notin [-1,+1], \quad x \in [a,b], \quad t \in (-1,+1),$$

where $m_z(x)$ is uniquely determined function.

In view of the Plemelj-Sokhotski formulas, the following equality is valid in the limiting values

$$\psi_t^+(x) + \psi_t^-(x) = \frac{1}{\pi i} \int_{-1}^{+1} \frac{\psi_s^+(x) - \psi_s^-(x)}{s-t} ds \tag{6}$$

$$x \in [a,b], \quad t \in (-1,+1).$$

Using (2), from (6) we are able to write

$$\overline{\psi}_t(x) + t \int_a^b \int_{-1}^{+1} \frac{sk(x,y)}{s-t} \overline{\psi}_t(y) dsdy + \int_a^b \int_{-1}^{+1} \frac{sk(x,y)}{s-t} \overline{\psi}_s(y) dsdy = \int_a^b k(x,y) f(y,t) dy, \tag{7}$$

where $\overline{\psi}_t(x) = \psi_t^+(x) - \psi_t^-(x)$.

Consider now the following equation

$$u(x,y) + \int_a^b \int_{-1}^{+1} \frac{tk(x,y)}{s-t} dsu(y,t)dy = \int_{-1}^{+1} \frac{s\overline{\psi}_s(x)}{t-s} ds + f(x,t) , \tag{8}$$

$$x \in [a,b], \quad t \in (-1,+1),$$

which has a unique solution.

Denote by

$$p(x,t) = \overline{\psi}_t(x) - \int_a^b k(x,y) u(y,t) dy ,$$

$$x \in [a,b], \quad t \in (-1,+1).$$

In view of (7) from (8) it follows that

$$p(x,t) + \int_a^b \int_{-1}^{+1} \frac{tk(x,y)}{s-t} p(y,t) dsdy = 0 .$$

But Ω_z has no eigenvalues on $[-1,+1]$; consequently, we have

$$\overline{\psi}_t(x) = \int_a^b k(x,y)u(y,t)dy$$

Taking into account the last assertion, from (8) we obtain

$$u(x,t) + \int_a^b \int_{-1}^{+1} \frac{tk(x,y)}{s-t} u(y,t)dsdy = \int_a^b \int_{-1}^{+1} \frac{sk(x,y)}{t-s} u(y,s)dsdy + f(x,t),$$

$$x \in [a,b],\ t \in (-1,+1),$$

which means that (4) holds and the proof is complete.

Now, the aim is to study the singular operator L more deeply. We wish to find the inverse operator of L. For this purpose, in the class D^* we need to introduce the following singular operator

$$(Su)(x,t) \equiv \int_a^b \int_{-1}^{+1} \frac{tk(x,y)}{t-s} u(y,s)dsdy + u(x,t) + \int_a^b \int_{-1}^{+1} \frac{tk(x,y)}{s-t} u(y,t)dsdy,$$

$$x \in [a,b],\ t \in (-1,+1).$$

We shall note the following property of the introduced operator. For any two functions u and v from D^*

$$\int_a^b \int_{-1}^{+1} uSvdtdx = \int_a^b \int_{-1}^{+1} vLudtdx.$$

Consequently, if equation (1) has a solution, then necessarily

$$\int_a^b \int_{-1}^{+1} vfdtdx = 0,$$

where v is any solution of the homogeneous equation

$$Sv = 0.$$

In our further consideration we will need also the following identity:

$$\int_a^b \int_{-1}^{+1} \frac{sk(x,y)}{t-s} \int_a^b \int_{-1}^{+1} \frac{s'k(y,y')}{s'-s} u(y',s')ds'dy'dsdy$$

$$= \int_a^b \int_{-1}^{+1} \frac{u(y',s')}{s'-t} \left(\int_a^b \int_{-1}^{+1} \frac{sk(x,y)s'k(y,y')}{t-s} dsdy - \int_a^b \int_{-1}^{+1} \frac{sk(x,y)s'k(y,y')}{s'-s} dsdy \right) ds'dy'$$

$$+\pi^2 t^2 \int_a^b \int_a^b k(x,y')k(y,y')u(y',t)dydy',$$

$$x \in [a,b],\ t \in (-1,+1).$$

This identity can be obtained by using the Bertrand-Poincare formula [3].

We can show the important property of the operator K introduced below:

$$(Ku)(x,t) = u(x,t) + \int_a^b r(x,y,t)u(y,t)dy,$$

$$x \in [a,b],\ t \in (-1,+1),$$

where

$$r(x,y,t)=\pi^2t^2\int_a^b k(x,y')k(y',y)dy' - \int_{-1}^{+1}\frac{tk(x,y)}{t-s}ds - \int_{-1}^{+1}\frac{tk(y,x)}{t-s}ds$$

$$+\int_a^b\int_{-1}^{+1}\frac{tk(x,y')}{t-s}ds\int_{-1}^{+1}\frac{tk(y'y)}{t-s'}ds'dy'.$$

It is easily seen that if $k(x,y)=k(y,x)$, then the following decomposition exists

$$Ku=(B+i\pi C)(B-i\pi C)u,$$

where

$$(Bu)(x,t)\equiv u(x,t)+\int_a^b\int_{-1}^{+1}\frac{tk(x,y)}{t-s}u(y,t)dsdy$$

$$(Cu)(x,t)\equiv t\int_a^b k(x,y)u(y,t)dt.$$

In general case also, the similar decomposition is true. It follows that one-dimensional regular integral equation

$$h(x,y,t)+\int_a^b r(y',y,t)h(x,y',t)dy'=r(x,y,t)\,,\quad x,y\in[a,b],\quad t\in(-1,+1)$$

admits unique solution. Now we can define the following integral operator

$$t(Tu)(x,t)=(S\dot{u})(x,t)+\int_a^b h(x,y,t)(S\dot{u})(y,t)dy\,,$$

$$x\in[a,b],\ \ t\in(-1,+1),$$

where $\dot{u}(x,t)=tu(x,t)$. It is not difficult to prove the following lemma.

Lemma. *The equality*

$$TLu=u$$

holds.

In conclusion, from the preceding consideration, the following theorem follows.

Main Theorem. *The equation*

$$Lu=f$$

is solvable if and only if $f\in D^*$ *satisfies the conditions*

$$\int_a^b\int_{-1}^{+1}t\tau_{z_k}^*(x,t)f(x,t)dtdx=0\,,\ z_k\in\Re\,.$$

If these conditions are fulfilled, then the solution is unique and it can be expressed by the formula

$$u=Tf.$$

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