ᲡᲐᲥᲐᲠᲗᲕᲔᲚᲝᲡ ᲛᲔᲪᲜᲘᲔᲠᲔᲑᲐᲗᲐ ᲔᲠᲝᲕᲜᲣᲚᲘ ᲐᲙᲐᲓᲔᲛᲘᲘᲡ ᲛᲝᲐᲛᲑᲔ, ტ. 9, №1, 2015
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 9, no. 1, 2015

Physics

Effective Potentials in the Reduced Alt-Grassberger-Sandhas-Khelashvili (AGSK) Equations and the Multi-Channel Problem

Anzor Khelashvili* and Teimuraz Nadareishvili**

* *Academy Member, Institute of High Energy Physics, I. Javakhishvili Tbilisi State University, Tbilisi*
St. Andrew the First-called Georgian University of Patriarchate of Georgia, Tbilisi
***Institute of High Energy Physics, I. Javakhishvili Tbilisi State University, Tbilisi*

ABSTRACT: There is such a formulation of integral equations for 3-body problem when scattering operators obey 3-dimensional equations (instead of 6-dimensional), and effective potentials are to be found from the Faddeev-like equations. In this formulation various approximate methods are developed, which have many advantages for studying the relevant problems. This approach is dominant for the last 50 years. The main problem consists in choosing the separable part of 2-body amplitudes in the kernels of 3-body equations. In the article we use the spectral representations for Green functions and show that all positive technical properties of AGSK equations remain and at the same time effective potentials should be simplified. We considered the effective potential in reduced 3-body problem. Potentials do not contain 2-particle bound state singularities in the 3-body space. Hence, wave functions have the same phases as scattering amplitudes. It follows that after performing angular decomposition in the final equations, 2-particle phases cancel each other and the effective potentials become real functions of arguments. Non-triviality of the obtained result consists in the following: the kernels of the equation for effective potentials do not depend on energy parameter, moreover one of the momenta rests on the energy-shell. Therefore, from 3-particle problem we need only 2-particle bound state wave functions and semi energy-shell amplitudes. This fact is somehow unexpected and may become an important thing for further advance in many particle dynamics. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key Words: *AGSK equations, pire potentials, off-energy shell.*

1. Introduction

An important property of Faddeev equations [1] is that they contain 2-body scattering matrices $\hat{T}_\alpha(z)$, but not pair potentials. However the Faddeev equations have one practical lack: their components have a complicated relation to the observable physical amplitudes.

Instead, there are the Lovelace equations [2] for $U^{(\pm)}_{\beta\alpha}(z)$ operators. They have the following properties: Kernel has the same structure, as Faddeev's one. Therefore, the equations are of Fredholm-Schmidt type.

Moreover, we need 2 sets of amplitudes $U^{(\pm)}_{\beta\alpha}(z)$, which are connected to the continuation behind the energetic surface.

It is well known, that the formulation of the 3-body problem is more convenient in terms of the Alt-Grassberger- Sandhas [3] and Khelashvili (AGSK) [4] operators, which give the equations, that are free from the abovementioned inconsistences. Below we consider the reduction of these equations to the three-dimensional ones and construct the many-channel effective potentials.

2. Equations for AGSK Operators

Let us introduce $A_{\alpha\beta}$ operators in the following symmetrical manner [4]:

$$G(z) = \delta_{\beta\alpha} G_\alpha(z) - G_\beta(z) A_{\beta\alpha}(z) G_\alpha(z), \tag{1}$$

(there is no summation in the repeated indices).

By this definition various channel amplitudes are related to the matrix elements of $A_{\alpha\beta}$ directly. For example [4]:

$$\left(\psi^{(-)}_{\beta n}, \psi^{(+)}_{\alpha m}\right) = \delta_{\beta\alpha}\delta_{nm} - 2\pi i \delta\left(E_{\beta n} - E_{\alpha m}\right)\left(\Phi_{\beta n}, A_{\beta\alpha}\left(E_{\alpha m} + i0\right)\Phi_{\alpha m}\right), \tag{2}$$

and so on for all the other amplitudes.

The introduced operators satisfy the following equations [4]

$$A_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha} G_0^{-1}(z) - \sum_\delta \bar{\delta}_{\alpha\delta} A_{\beta\delta}(z) G_0(z) \hat{T}_\delta(z) \tag{3}$$

and

$$A_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha} G_0^{-1}(z) - \sum_\gamma \bar{\delta}_{\beta\gamma} \hat{T}_\gamma(z) G_0(z) A_{\gamma\delta}(z). \tag{4}$$

They are connected to the Lovelace operators by

$$A_{\beta\alpha}(z) = U^{(+)}_{\beta\alpha}(z) - \bar{\delta}_{\beta\alpha} G_\alpha^{-1}(z) = U^{(-)}_{\beta\alpha}(z) - \bar{\delta}_{\beta\alpha} G_\beta^{-1}(z). \tag{5}$$

The equations obtained have the following positive features:

- All the advantages of the Faddeev equations are retained. Particularly, the symmetry among them, as they consist only of 2-body amplitudes, $\hat{T}_\alpha(z)$.
- They are related to channel amplitudes in a simple manner.
- Indices take values 0,1,2,3, i.e. these equations take into account all the channels in 3-body system.
- There are 16 operators, but not 32, as in Lovelace case.

3. Reduction of AGSK Equations, Effective Potentials

It is well known that 2-body scattering operator can be always decomposed into factorizable (separable) and remainder parts:

$$\hat{T}_\gamma(z) = \hat{T}^f_\gamma(z) + \hat{T}^R_\gamma(z) \tag{6}$$

Naturally, this decomposition is not unique, but in any problem it may be achieved. After such a decomposition AGSK equations split into two systems [4] :

$$A_{\beta\alpha}(z) = B_{\beta\alpha}(z) - \sum_\gamma B_{\beta\gamma}(z) G_0(z) \hat{T}^f_\gamma(z) G_0(z) A_{\gamma\alpha}(z) \tag{7}$$

and

$$B_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha} G_0^{-1}(z) - \sum_{\gamma} \bar{\delta}_{\beta\gamma} \hat{T}_{\gamma}^{R}(z) G_0(z) B_{\gamma\alpha}(z). \tag{8}$$

We see that the first one (7) is a system of 3-dimensional equations, in which $B_{\gamma\alpha}(z)$ are operators of ***"effective potentials***". On the other hand, they must be constructed from the equations which have the same complexity.

These forms of equations dictate a rather simple scheme of approximate calculations: potential operators can be constructed from the second equations in needed approximation and then use them in the first equations, which are 3-dimensional.

This is the *first* step of reduction. The followed step is connected with the special choice of a separable term. We note that usually most authors were restricted only by separable parts in their calculations.

Below we want to move further. In particular, Karlson and Zeiger [5] used a special form for separable parts, basing on the spectral representation of 2-body Green function:

$$G_{\gamma}(z) = \sum_{r} \int \frac{\left|\Phi_{\gamma r}, \boldsymbol{q}_{\gamma}''\right\rangle d\boldsymbol{q}_{\gamma}'' \left\langle \Phi_{\gamma r}, \boldsymbol{q}_{\gamma}''\right|}{\tilde{q}_{\gamma}''^{2} - E_{\gamma r} - z} + \int \frac{\left|\psi_{\boldsymbol{p}_{\gamma}''}^{\gamma} \boldsymbol{q}_{\gamma}''\right\rangle d\boldsymbol{p}_{\gamma}'' d\boldsymbol{q}_{\gamma}'' \left\langle \psi_{\boldsymbol{p}_{\gamma}''}^{\gamma} \boldsymbol{q}''\right|}{\tilde{q}_{\gamma}''^{2} + \tilde{p}_{\gamma}''^{2} - z^{\gamma}} \equiv G_{\gamma}^{B}(z) + G^{C}(z). \tag{9}$$

Here $\Phi_{\gamma r}$ is a bound state wave function for γ channel with energy $E_{\gamma r}$ (discrete spectrum) and $\psi_{\boldsymbol{p}''}^{\gamma}$ is a wave function of the continuous spectrum in the same channel. Here, only the completeness relation is explored in the 2-body sub-systems. Let us take this representation into account in the definition of the 2-body amplitudes

$$\hat{T}_{\gamma}(z) G_0(z) = V_{\gamma} G_{\gamma}(z) \tag{10}$$

and take bound state parts as a separable contribution

$$\hat{T}_{\gamma}^{f}(z) G_0(z) = V_{\gamma} G_{\gamma}^{B}(z) \tag{11}$$

and consider the continuous contribution by the remainder relation

$$\hat{T}_{\gamma}^{R}(z) G_0(z) = V_{\gamma} G_{\gamma}^{C}(z). \tag{12}$$

If we use the relation

$$G_{\gamma}(z) = G_0(z) + G_0(z) \hat{T}_{\gamma}(z) G_0(z), \tag{13}$$

it follows

$$G_0(z) \hat{T}_{\gamma}^{f}(z) G_0(z) = G_{\gamma}^{B}(z). \tag{14}$$

This is a crucial moment, on validity of which our further consideration is dependent. In this respect our equations become:

$$A_{\beta\alpha}(z) = B_{\beta\alpha}(z) + \sum_{\gamma} B_{\beta\gamma}(z) \tilde{G}_{\gamma}^{B}(z) A_{\gamma\alpha}(z), \tag{15}$$

where

$$\tilde{G}_{\gamma}^{B}(z) = G_0(z) V_{\gamma} G_{\gamma}^{B}(z) = G_{\gamma}^{B}(z) V_{\gamma} G_0(z), \tag{16}$$

and

$$B_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha} G_0^{-1}(z) - \sum_{\gamma} \bar{\delta}_{\beta\gamma} V_{\gamma} G_{\gamma}^{C}(z) B_{\gamma\alpha}(z). \tag{17}$$

From (15) 3-dimensional equations for all physical channels follow.

Very non-trivial result follows for potential matrices. After substitution of the spectral representation we derive the following intermediate relations:

- Lippmann-Schwinger-like equation

$$V_\gamma \left| \boldsymbol{p}''_\gamma \boldsymbol{q}''_\gamma \right\rangle = \hat{T}_\gamma \left(\tilde{p}''^2 + i0 \right) \left| \boldsymbol{p}''_\gamma \boldsymbol{q}''_\gamma \right\rangle \tag{18}$$

and

$$B_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha} G_0^{-1}(z) - \sum_\gamma \bar{\delta}_{\beta\gamma} \int \frac{\hat{T}_\gamma \left(\tilde{p}''^2 + i0 \right) \left| \boldsymbol{p}''_\gamma \boldsymbol{q}''_\gamma \right\rangle d\boldsymbol{p}''_\gamma d\boldsymbol{q}''_\gamma \left\langle \psi^\gamma_{\boldsymbol{p}''_\gamma} \boldsymbol{q}''_\gamma \right| B_{\gamma\alpha}(z)}{q''^2_\gamma + \tilde{p}''^2_\gamma - z}. \tag{19}$$

Here, entering the kernel $\hat{T}_\gamma$ does not depend on z. Moreover one of the momenta $\boldsymbol{p}''_\lambda$ rests on the energy shell.

Therefore, we need the following things from 2-particle problem:

- the bound-state wave functions and
- semi-energetic scattering amplitudes.

Let us rewrite derived equations in the explicit form:

$$T_{\beta m,\alpha n}\left(\boldsymbol{q}'_\beta, \boldsymbol{q}_\alpha; z \right) = V_{\beta m,\alpha n}\left(\boldsymbol{q}'_\beta, \boldsymbol{q}_\alpha; z \right) + \sum_{\delta r} \int d\boldsymbol{q}''_\delta \frac{V_{\beta m,\delta r}\left(\boldsymbol{q}'_\beta, \boldsymbol{q}''_\delta; z \right)}{E_{\delta r} + z - \tilde{\boldsymbol{q}}''^2_\delta} T_{\delta r,\alpha n}\left(\boldsymbol{q}''_\delta, \boldsymbol{q}_\alpha; z \right), \tag{20}$$

where the potentials are:

$$V_{\beta m,\alpha n}\left(\boldsymbol{q}'_\beta, \boldsymbol{q}_\alpha; z \right) = \int d\boldsymbol{p}'_\beta \Phi^*_{\beta m}\left(\boldsymbol{p}'_\beta \right) \left\langle \boldsymbol{p}'_\beta \boldsymbol{q}'_\beta \right| B_{\beta\alpha}(z) \left| \boldsymbol{p}_\alpha \boldsymbol{q}_\alpha \right\rangle \Phi_{\alpha n}\left(\boldsymbol{p}_\alpha \right) d\boldsymbol{p}_\alpha \tag{21}$$

Hence, the problem requires knowledge of the following quantities:

$$\left\langle \Phi_{\beta m} \boldsymbol{q}'_\beta \right| \hat{T}_\gamma \left(\tilde{p}''^2_\gamma \right) \left| \boldsymbol{p}''_\gamma \boldsymbol{q}''_\gamma \right\rangle = \int d\boldsymbol{p}'_\beta \Phi^*_{\beta m}\left(\boldsymbol{p}'_\beta \right) \left\langle \boldsymbol{p}'_\beta \boldsymbol{q}'_\beta \right| \hat{T}_\gamma \left(\tilde{p}''^2_\gamma \right) \left| \boldsymbol{p}''_\gamma \boldsymbol{q}''_\gamma \right\rangle, \tag{22}$$

$$\left\langle \Phi_{\beta m} \boldsymbol{q}'_\beta \right| \hat{T}_\gamma \left(\tilde{p}''^2_\gamma + i0 \right) \left| \boldsymbol{p}''_\gamma \boldsymbol{q}''_\gamma \right\rangle = \Phi^*_{\beta m}\left(\boldsymbol{q}''_\gamma + \frac{m_\gamma}{m_\alpha + m_\gamma} \boldsymbol{q}'_\beta \right) t_\gamma \left(-\boldsymbol{q}'_\beta - \frac{m_\beta}{m_\alpha + m_\beta} \boldsymbol{q}''_\gamma, \boldsymbol{p}''_\gamma; \tilde{p}''^2_\gamma + i0 \right) \tag{23}$$

and

$$\left\langle \psi^\beta_{\boldsymbol{p}'_\beta} \boldsymbol{q}'_\beta \right| \hat{T}_\gamma \left(\tilde{p}''^2_\gamma + i0 \right) \left| \boldsymbol{p}''_\gamma \boldsymbol{q}''_\gamma \right\rangle = \psi^{*\beta}_{\boldsymbol{p}'_\beta}\left(\boldsymbol{q}''_\gamma + \frac{m_\gamma}{m_\alpha + m_\gamma} \boldsymbol{q}'_\beta \right) t_\gamma \left(-\boldsymbol{q}'_\beta - \frac{m_\beta}{m_\alpha + m_\beta} \boldsymbol{q}''_\gamma, \boldsymbol{p}''_\gamma; \tilde{p}''^2_\gamma + i0 \right). \tag{24}$$

Other necessary needed matrix elements may be written analogously in the explicit form.

6. Conclusions

We considered the effective potential in the reduced 3-body problem. It was established above that in the final equations we have the following quantities:

1. 2-body physical (on- energy- shell) amplitudes,
2. Potentials do not contain 2-particle bound-state singularities in the 3-body space. Therefore, the wave functions have the same phases, as scattering amplitudes.
3. When we perform the angular decomposition, 2-particle phases are cancelled and effective potentials become real functions.

Non-triviality of the obtained result consists in the following: The kernels of the equation for effective potentials do not depend on the energy parameter z, moreover one of the momenta $\left(p''_{\gamma}\right)$ rests on the energy-shell. Therefore, from 3-particle problem we need only the 2-particle bound-state wave functions and semi-energy-shell amplitudes. This fact is somehow unexpected and may become an important fact for further advance in multi-particle dynamics.

Acknowledgement. This work is financially supported *by Rustaveli Foundation* (Project D1/13/02). The authors are grateful for support.

ფიზიკა

ეფექტური პოტენციალები დაყვანილ ალტ-გრასბერგერ-სანდჰას-ხელაშვილის (აგსხ) განტოლებებში და მრავალი არხის პრობლემა

ა. ხელაშვილი*, თ. ნადარეიშვილი**

** აკადემიის წევრი, ი. ჯავახიშვილის თბილისის სახელმწიფო უნივერსიტეტის მაღალი ენერგიების ფიზიკის ინსტიტუტი, თბილისი*
საქართველოს საპატრიარქოს წმიდა ანდრია პირველწოდებულის ქართული უნივერსიტეტი
***ი. ჯავახიშვილის თბილისის სახელმწიფო უნივერსიტეტის მაღალი ენერგიების ფიზიკის ინსტიტუტი, თბილისი*

ცნობილია აგსხ ოპერატორებისათვის ინტეგრალური განტოლებების ისეთი ფორმულირება, როდესაც ისინი აკმაყოფილებენ 3-განზომილებიან განტოლებებს (ნაცვლად 6-განზომილებიანისა) და ეფექტური პოტენციალები შეიძლება ნაპოვნი იქნეს ფადეევის მსგავსი განტოლებებიდან. ამ ფორმულირებაში დამუშავებულია სხვადასხვა მიახლოებითი მეთოდი და მას აქვს მრავალი უპირატესობა შესაბამისი პრობლემების შესწავლისას. ეს მიდგომა დომინირებს ბოლო 50 წლის განმავლობაში. ძირითად პრობლემას წარმოადგენს სეპარაბელური 2-ნაწილაკოვანი ამპლიტუდების შერჩევა 3-ნაწილაკოვან განტოლებებში. მოცემულ სტატიაში ჩვენ ვიყენებთ გრინის ფუნქციების სპექტრალურ წარმოდგენას და ვაჩვენებთ, რომ ყველა პოზიტიური თვისება აგსხ განტოლებებისა ძალაში რჩება და ამავდროულად ეფექტური პოტენციალები მარტივდება. პოტენციალები არ შეიცავენ 2-ნაწილაკოვანი ბმული მდგომარეობების სინგულარობებს 3-ნაწილაკოვან სივრცეში. შესაბამისად, ტალღურ ფუნქციებს იგივე ფაზები გააჩნიათ რაც გაფანტვის ამპლიტუდებს. აქედან გამომდინარე, კუთხური ცვლადებით განცალებისას საბოლოო განტოლებაში 2-ნაწილაკოვანი ფაზები ერთმანეთს აბათილებენ და ეფექტური პოტენციალები არგუმენტის რეალური ფუნქციები ხდება. მიღებული შედეგების არატრივიალურობა შემდეგში მდგომარეობს: ეფექტური პოტენციალების განტოლების გულები არ არის დამოკიდებული ენერგიის პარამეტრზე და გარდა ამისა, ერთ-ერთი იმპულსი ძევს ენერგეტიკულ ზედაპირზე. ამიტომ სამი ნაწილაკის პრობლემაში ჩვენ გვჭირდება მხოლოდ 2-ნაწილაკოვანი ბმული მდგომარეობების ტალღური ფუნქციები და ამპლიტუდები ნახევრად ენერგეტიკულ ზედაპირზე. ეს საკმაოდ მოულოდნელი ფაქტია და შესაძლოა მნიშვნელოვანი გახდეს მრავალი ნაწილაკის დინამიკის კვლევისას.

REFERENCES

1. *Faddeev L.D.* (1960) Sov. J. JETF., **39**: 1459.
2. *Lovelace C.* (1964) Phys. Rev., **135B**: 1225.
3. *Alt E.O., Grassberger P., Sandhas W.* (1967) Nucl. Phys., **82**: 167.
4. *Khelashvili A.A.* (1967) Preprint of Joint Institute of Nuclear Researches. Dubna **P2**: P3371 (in Russian).
5. *Karlson B.R., Zeiger E.M.* (1974) Phys. Rev., **D 9**: 1761; **D10**: 129.

Received February, 2015