Mathematics

On Topologically Finite Spaces

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ABSTRACT. The notion of topologically finite space was introduced in 2012. In the present paper the question whether the union of finite number of topologically finite spaces is topologically finite is studied. More precisely, it is shown that the union of two topologically finite spaces (even in the realm of separable, metrizable and connected spaces) may not be topologically finite. *© 2015 Bull. Georg. Natl. Acad. Sci.*

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Introduction

The notion of topologically finite space was introduced in [1] (see definition below), where, in particular, it was shown that for any $n \in \{0,1,2,...\} \cup \{\infty\}$ there exists a separable and metrizable space X_n with $\dim X_n = n$ and $|X_n| = \boldsymbol{c}$ (here, as usual, $\dim$ stands for the classical covering dimension function, $|X_n|$ denotes cardinality of X_n and $\boldsymbol{c}$ is the cardinality of continuum). In the same paper it was established also, that if a topological space is the union of its two disjoint, topologically finite, open and connected subspaces, then the space itself is topologically finite. The following question naturally rises, whether the union of any two topologically finite spaces is topologically finite. Below we show that there exists a topologically infinite locally compact and path connected subspace of the real plane which is the union of its two topologically finite locally compact and path connected subspaces.

Definitions and Notations

A topological space X is topologically finite provided there is no proper subspace of X which is homeomorphic to the whole X. Otherwise we say that the space X is topologically infinite. Evidently, the notion of topological finiteness is topological invariant.

Basically, we accept notations and definitions given in [1].

$\mathbb{R}^2$ denotes the real plane with natural topology. $\mathbb{Z}$ is the set of all integers and $\mathbb{N}$ is the set of all natural numbers.

For any $(a_1,b_1),\ (a_2,b_2)\in\mathbb{R}^2$, $\|(a_1,b_1)-(a_2,b_2)\|=\sqrt{(a_1-a_2)^2-(b_1-b_2)^2}$.

$\mathbb{Z}_{odd}$ (respectively, $\mathbb{Z}_{even}$) denotes the set of all odd (respectively, even) integers.

For any $n\in\mathbb{Z}$ denote: $S_n=\left\{(x,y)\in\mathbb{R}^2:\left\|(x,y)-\left(n,\frac{1}{2}\right)\right\|=\frac{1}{2}\right\}$. Thus, each S_n is the circumference of the circle with the center at $\left(n,\frac{1}{2}\right)$ and radius $\frac{1}{2}$.

By X we denote the union of all S_n-s: $X=\bigcup_{n\in\mathbb{Z}}S_n$. It is obvious that X is the locally compact and path connected subspace of $\mathbb{R}^2$.

Let $\gamma_n=\left(n,\frac{1}{2}\right)$ for each $n\in\mathbb{Z}$. Thus, γ_n is the point at which S_n touches S_{n+1}.

Denote: $X_1=\bigcup_{n\in\mathbb{Z}}\{\gamma_k\}$ and $X_2=X\setminus X_1$. Evidently, $X=X_1\bigcup X_2$ and $X_1\bigcap X_2=\varnothing$.

We say, that γ_i and γ_j are neighbor points of X_1 (or simply, neighbor points), if $|i-j|=1$. Obviously, γ_i and γ_j are neighbor points of X_1 if and only if there exists $n\in\mathbb{Z}$ such that $\gamma_i,\gamma_j\in S_n$.

Let $pr_1:\mathbb{R}^2\to\mathbb{R}$ be the first projection, i.e., for any $\alpha=(x,y)\in\mathbb{R}^2$, $pr_1(\alpha)=x$. It is clear that for each $k\in\mathbb{Z}$, $pr_1(\gamma_k)=k$.

For any $\alpha,\beta\in X$, where $pr_1(\alpha)<pr_1(\beta)$ denote: $X_1^{\alpha<\beta}=\{k\in\mathbb{Z}|\,pr_1(\alpha)<k<pr_1(\beta)\}$. Note that $X_1^{\alpha<\beta}=\varnothing$, if and only if there exists $i\in\mathbb{Z}$ with $\alpha,\beta\in S_i$.

For any $n\in\mathbb{Z}$ and any distinct points $\alpha,\beta\in S_n$ there are exactly two (closed) arcs C_1 and C_2 in S_n with end-points α and β. The sets $C_1\setminus\{\alpha,\beta\}$ and $C_2\setminus\{\alpha,\beta\}$ are called below open arcs in S_n with end-points α and β.

Symbol $\square$ denotes the end of proof.

Main Construction

We begin with the following trivial assertion.

Assertion. Let $\varphi:A\to B$ be an injective mapping from a set A to a set B. Let also, $A_1,A_2\subset A$ and $A_1\bigcap A_2$ consists of exactly n points of A: $A_1\bigcap A_2=\{a_1,...,a_n\}$. Then $\varphi(A_1)\bigcap\varphi(A_2)=\{\varphi(a_1),...,\varphi(a_n)\}$, i.e., the set $\varphi(A_1)\bigcap\varphi(A_2)$ consists of exactly n points of B as well.

Proposition 1. Let $\alpha,\beta\in X$ be two distinct points of X. Then if $\alpha,\beta\in S_n$ for some $n\in\mathbb{Z}$, there exist two connected subsets C_1 and C_2 of X such, that $C_1\cap C_2=\{\alpha,\beta\}$. Furthermore, let for any $i\in\mathbb{Z}$ we have $\{\alpha,\beta\}\not\subset S_i$. Let also, $pr_1(\alpha)<pr_1(\beta)$ and C is a connected subset of X with $\alpha,\beta\in C$. Then for any $k\in X_1^{\alpha<\beta}$ we shall have: $a\neq\gamma_k$, $\beta\neq\gamma_k$ and $\gamma_k\in C$ (as we have already noted above, in this case $X_1^{\alpha<\beta}\neq\varnothing$).

Proof. Suppose first, $\alpha,\beta\in S_n$ for some $n\in\mathbb{Z}$. Let C_1 and C_2 be two distinct (closed) arcs in S_n with end-points α and β. Clearly, C_1 and C_2 are connected subsets of X and $C_1\bigcap C_2=\{\alpha,\beta\}$.

Suppose now, for any $i\in\mathbb{Z}$ we have $\{\alpha,\beta\}\not\subset S_i$, $pr_1(\alpha)<pr_1(\beta)$ and C is a connected subset of X with $\alpha,\beta\in C$ and let k be any element of $X_1^{\alpha<\beta}\neq\varnothing$.

It is obvious that $a\neq\gamma_k$ and $\beta\neq\gamma_k$,. Show that $\gamma_k\in C$. Indeed, assume, $\gamma_k\notin C$. Denote:

$$U=C\bigcap\left(\left(\bigcup_{i\leq k}S_i\right)\setminus\{\gamma_k\}\right),\quad V=C\bigcap\left(\left(\bigcup_{i>k}S_i\right)\setminus\{\gamma_k\}\right).$$

It is obvious that $U\neq\varnothing$ ($\alpha\in U$), $V\neq\varnothing$ ($\beta\in V$), $U\bigcap V=\varnothing$, U and V are open subsets of C and (since by the assumption $\gamma_k \notin C$) $C = U \bigcup V$. We get a contradiction, because C is connected. □

Proposition 2. For any homeomorphic embedding $h: X \to X$ we have: $h(X_1) \subset X_1$.

Proof. Suppose $h(X_1) \bigcap X_2 \neq \varnothing$. Then there has to be $n \in \mathbb{Z}$ such that $h(\gamma_n) \in X_2$. Since $h(\gamma_n) \in X_2$, there is unique $k \in \mathbb{Z}$ with $h(\gamma_n) \in S_k$.

Let $V \subset X$ be a connected neighborhood of the point γ_n in X. Then as h is homeomorphic embedding, $U = h(V)$ is a connected neighborhood of $h(\gamma_n)$ in $h(X)$.

Two cases are possible: (a) $S_k \subset h(X)$; (b) $S_k \not\subset h(X)$.

Case (a) ($S_k \subset h(X)$). Obviously, any neighborhood of γ_n in S_n (and in particular, V) intersects $S_n \setminus \{\gamma_n\}$ as well as $S_{n+1} \setminus \{\gamma_n\}$. Let $\alpha_0 \in V \bigcap (S_n \setminus \{\gamma_n\})$ and $\beta_0 \in V \bigcap (S_{n+1} \setminus \{\gamma_n\})$. Denote: $\alpha_0' = h(\alpha_0)$ and $\beta_0' = h(\beta_0)$. Clearly, $\alpha_0', \beta_0' \in U$ and $\alpha_0' \neq \beta_0'$.

Let C_1' and C_2' be two distinct arcs in S_k with end-points α_0' and β_0'. Evidently, $C_1', C_2' \subset h(X)$ and $C_1' \bigcap C_2' = \{\alpha_0', \beta_0'\}$.

Then (according to the Assertion above) we should have:

$h^{-1}(C_1') \bigcap h^{-1}(C_2') = \{h^{-1}(\alpha_0'), h^{-1}(\beta_0')\} = \{h^{-1}(h(\alpha_0)), h^{-1}(h(\beta_0))\} = \{\alpha_0, \beta_0\}$.

On the other hand, we have: $\alpha_0 \neq \gamma_0$, $\beta_0 \neq \gamma_0$, $pr_1(\alpha_0) < pr_1(\beta_0)$, there is no $i \in \mathbb{Z}$ with $\alpha_0, \beta_0 \in S_i$, $X_1^{\alpha_0 < \beta_0} = \{\gamma_n\}$ and $h^{-1}(C_1')$ and $h^{-1}(C_2')$ are connected subsets of X. Hence by Proposition 1, $h^{-1}(C_1') \bigcap h^{-1}(C_2') \supset \{\alpha_0, \gamma_n, \beta_0\}$, which (taking into the account the equality $h^{-1}(C_1') \bigcap h^{-1}(C_2') = \{\alpha_0, \beta_0\}$) leads to a contradiction.

Case (b) ($S_k \not\subset h(X)$). If $S_k \not\subset h(X)$, there has to exist at least one point $\sigma' \in S_k$ with $\sigma' \notin h(X) \supset U$.

Obviously, any neighborhood of γ_n in S_n (and in particular, V) intersects $S_n \setminus \{\gamma_n\}$. Let $\delta_0 \in V \bigcap (S_n \setminus \{\gamma_n\})$. Denote: $\delta_0' = h(\delta_0)$. Clearly, $\delta_0' \in U$ and $\delta_0' \neq h(\gamma_n)$. Let C' be an open arc in S_k with end-points δ_0' and $h(\gamma_n)$ such that $\sigma' \notin C'$.

We shall now show that any connected subset $C^* \subset h(X) \subset X$, containing the points δ_0' and $h(\gamma_n)$, contains the whole C'. Indeed, assume, there is a point $\varsigma' \in C'$ such that $\varsigma' \notin C^*$. Note that since $\sigma' \notin h(X) \supset C^*$ then $\sigma' \notin C^*$. Obviously, there are two open subsets W_1 and W_2 of the subspace $X \setminus \{\sigma', \varsigma'\}$ of X such that $W_1 \bigcap W_2 = \varnothing$, $W_1 \bigcup W_2 = X \setminus \{\sigma', \varsigma'\}$, $\delta_0' \in W_1$ and $h(\gamma_n) \in W_2$. Denote: $W_1' = W_1 \bigcap C^*$ and $W_2' = W_2 \bigcap C^*$. Then W_1' and W_2' are open subsets of C^*, both nonempty ($\delta_0' \in W_1'$ and $h(\gamma_n) \in W_1'$), $W_1' \bigcap W_2' = \varnothing$ and $W_1' \bigcup W_2' = C^*$. Thus we get a contradiction (because the subspace $C^* \subset h(X) \subset X$ is connected).

Let now C_1 and C_2 be two distinct arcs in S_n with end-points δ_0 and γ_n. Since $C_1 \bigcap C_2 = \{\delta_0, \gamma_n\}$, by the Assertion above we have: $h(C_1) \bigcap h(C_2) = \{h(\delta_0), h(\gamma_n)\} = \{\delta_0', h(\gamma_n)\}$.

On the other hand, consider any point $\varsigma_0' \in C'$. It is clear, that $\varsigma_0' \neq \delta_0'$ and $\varsigma_0' \neq h(\gamma_n)$. Furthermore, $h(C_1)$ and $h(C_2)$ are connected subsets of $h(X) \subset X$ both containing the points δ_0' and $h(\gamma_n)$. Hence (as we have already seen) $\varsigma_0' \in h(C_1)$ and $\varsigma_0' \in h(C_2)$, i.e., $h(C_1) \bigcap h(C_2) \supset \{\delta_0', \varsigma_0', h(\gamma_n)\}$, which contradicts the equality $h(C_1) \bigcap h(C_2) = \{\delta_0', \eta_0'\}$. □

Proposition 3. Any homeomorphic embedding $h: X \to X$ sends neighbor points of X_1 to the neighbor points of X_1.

Proof. Take any $n \in \mathbb{Z}$. As (by Proposition 2) $h(X_1) \subset X_1$, there are $k, m \in \mathbb{Z}$ with $h(\gamma_n) = \gamma_k$ and $h(\gamma_{n-1}) = \gamma_m$. We have to show, that either $m = k-1$ or $m = k+1$. Assume on the contrary, $m \neq k-1$ and $m \neq k+1$. Note that as h is injection, $h(\gamma_{n-1}) = \gamma_m$, $h(\gamma_n) = \gamma_k$ and $\gamma_{n-1} \neq \gamma_n$, then $m \neq k$. Thus, only two cases are possible: (a) $m < k-1$; (b) $m > k+1$.

Case (a) ($m < k-1$). The points γ_{n-1} and γ_n belong to S_n. Then, by Proposition 1, there are two connected subsets C_1 and C_2 of S_n (namely, the two (closed) arcs in S_n with end points γ_{n-1} and γ_n) such that $C_1 \cap C_2$ consists of exactly two points: $C_1 \cap C_2 = \{\gamma_{n-1}, \gamma_n\}$. Hence (since h is injection), by the Assertion we shall have: $h(C_1) \cap h(C_2) = \{h(\gamma_{n-1}), h(\gamma_n)\} = \{\gamma_m, \gamma_k\}$. That is, $h(C_1) \cap h(C_2)$ must consist of exactly two points as well.

Furthermore, since $m < k-1$, there is no $i \in \mathbb{Z}$, with $\gamma_m, \gamma_k \in S_i$. As $h(C_1)$ and $h(C_2)$ are both connected, $\gamma_m, \gamma_k \in h(C_1)$ and $\gamma_m, \gamma_k \in h(C_2)$, then by Proposition 1, there is at least one $j \in \mathbb{Z}$ such that $\gamma_j \neq \gamma_m$, $\gamma_j \neq \gamma_k$, $\gamma_j \in h(C_1)$ and $\gamma_j \in h(C_2)$; hence $\gamma_j \in h(C_1) \cap h(C_2)$. Consequently, on the one hand, $h(C_1) \cap h(C_2) = \{\gamma_m, \gamma_k\}$ and, on the other hand, we have: $h(C_1) \cap h(C_2) \supset \{\gamma_m, \gamma_k, \gamma_j\}$ (where $\gamma_j \neq \gamma_m$, $\gamma_j \neq \gamma_k$). Contradiction.

Consideration of Case (b) is carried out in the similar way. □

Proposition 4. For any homeomorphic embedding $h: X \to X$ we have: $h(X_1) = X_1$.

Proof. Consider any $\gamma_k \in X$. By proposition 2 there is $m \in \mathbb{Z}$ such that $h(\gamma_k) = \gamma_m$.

By Proposition 3, two cases are possible: (a) $h(\gamma_{k+1}) = \gamma_{m+1}$ and (b) $h(\gamma_{k+1}) = \gamma_{m-1}$.

Case (a). $h(\gamma_{k+1}) = \gamma_{m+1}$. We shall show that for any integer $i \in \mathbb{N}$ the following equalities (1) $h(\gamma_{k-i}) = \gamma_{m-i}$ and (2) $h(\gamma_{k+i}) = \gamma_{m+i}$ hold.

Proof of the equality (1). Let $i = 1$. Since γ_{k-1} and γ_k are neighbor points of X_1 and $h(\gamma_k) = \gamma_m$, by Proposition 3 we must have: $h(\gamma_{k-1}) = \gamma_{m-1}$ or $h(\gamma_{k-1}) = \gamma_{m+1}$. But, $h(\gamma_{k+1}) = \gamma_{m+1}$ and as h, in particular, is injective and $\gamma_{k-1} \neq \gamma_{k+1}$, the equality $h(\gamma_{k-1}) = \gamma_{m+1}$ is impossible. So, $h(\gamma_{k-1}) = \gamma_{m-1}$.

Assume now, (1) is true for any $1 \le j \le i$. Since γ_{k-i-1} and γ_{k-i} are neighbor points of X_1 and (by the assumption) $h(\gamma_{k-i}) = \gamma_{m-i}$, then by Proposition 3 we must have: $h(\gamma_{k-i-1}) = \gamma_{m-i-1}$ or $h(\gamma_{k-i-1}) = \gamma_{m-i+1}$. But, by the assumption $h(\gamma_{k-i+1}) = h(\gamma_{k-(i-1)}) = h(\gamma_{k-(i-1)}) = \gamma_{m-i+1}$ and as h is injective and $\gamma_{k-i-1} \neq \gamma_{k-i+1}$, the equality $h(\gamma_{k-i-1}) = \gamma_{m-i+1}$ is impossible. So, $h(\gamma_{k-i-1}) = \gamma_{m-i-1}$. The equality (1) is proved.

Proof of the equality (2) is carried out in the same way.

Case (b). $h(\gamma_{k+1}) = \gamma_{m-1}$. In the same way one can prove, that for any integer $i \in \mathbb{N}$ the following equalities take place: (3) $h(\gamma_{k-i}) = \gamma_{m+i}$ and (4) $h(\gamma_{k+i}) = \gamma_{m-i}$.

Now show that $h(X_1) = X_1$. According to the Proposition 2 it suffices to prove that $X_1 \subset h(X_1)$.

Take any $\gamma \in X_1$. Then either $\gamma = \gamma_m$ or there is $i \ge 1$ such that either $\gamma = \gamma_{m-i}$ or $\gamma = \gamma_{m+i}$.

Suppose the first Case (a) takes place. If $\gamma = \gamma_m$, then $\gamma = \gamma_m = h(\gamma_k)$. If $\gamma = \gamma_{m-i}$ (where $i \ge 1$) then, by the equality (1) $\gamma = \gamma_{m-i} = h(\gamma_{k-i})$. If $\gamma = \gamma_{m+i}$ (where $i \ge 1$) then, by the equality (2) $\gamma = \gamma_{m+i} = h(\gamma_{k+i})$.

Suppose now, Case (b) takes place. If $\gamma = \gamma_m$, then $\gamma = \gamma_m = h(\gamma_k)$. If $\gamma = \gamma_{m+i}$ (where $i \ge 1$) then, by the equality (3) $\gamma = \gamma_{m+i} = h(\gamma_{k-i})$. If $\gamma = \gamma_{m-i}$ (where $i \ge 1$) then, by the equality (4) $\gamma = \gamma_{m-i} = h(\gamma_{k+i})$.

Consequently, for any $\gamma \in X_1$ there exists $\gamma' \in X_1$ with $h(\gamma') = \gamma$ i.e., $X_1 \subset h(X_1)$. □

The following Corollary follows immediately from the proposition 4.

Corollary. For any homeomorphic embedding $h: X \to X$ we have: $h(X_2) \subset X_2$.

Theorem 1. The subspace X of $\mathbb{R}^2$ is topologically finite.

Proof. Assume, X is not topologically finite. Then there exists a homeomorphic embedding $h: X \to X$ with $h(X) \neq X$. Hence, there is a point $\xi_0 \in X \setminus h(X)$. According to Proposition 4, $X_1 = h(X_1) \subset h(X)$. Consequently, $\xi_0 \notin X_1$ and therefore $\xi_0 \in X_2$. Clearly, there is unique $k \in \mathbb{Z}$ such that $\xi_0 \in S_k$.

Let S_k^+ (respectively, S_k^-) be the "upper" (respectively, "lower") open arc in S_k with end-points γ_{k-1} and γ_k. More precisely,

$$S_k^+ = \left\{(x,y) \in S_k \mid y > \frac{1}{2}\right\} \text{ and } S_k^+ = \left\{(x,y) \in S_k \mid y < \frac{1}{2}\right\}.$$

Without loss of generality, assume $\xi_0 \in S_k^+$.

Now we shall show that any point of S_k^- belongs to $h(X)$. Indeed, suppose there is a point $\xi \in S_k^-$ which does not belong to $h(X)$. It is clear, that $X \setminus \{\xi_0, \xi\}$ can be represented as the union of two open subsets W_1 and W_2 of X with $\{..., \gamma_{k-2}, \gamma_{k-1}\} \subset W_1$, $\{\gamma_k, \gamma_{k+1}, ...\} \subset W_2$, $W_1 \cap W_2 = \varnothing$ and $W_1 \cup W_2 = X \setminus \{\xi_0, \xi\}$. As, $\xi_0 \notin h(X)$ and (by the assumption) $\xi \notin h(X)$, then $h(X) = (W_1 \cap h(X)) \cup (W_2 \cap h(X))$. Evidently, $W_1 \cap h(X)$ and $W_2 \cap h(X)$ are disjoint, open subsets of $h(X)$ and $h(X) = (W_1 \cap h(X)) \cup (W_2 \cap h(X))$. Besides (as $\{..., \gamma_{k-2}, \gamma_{k-1}\} \subset W_1 \cap h(X)$ and $\{\gamma_k, \gamma_{k+1}, ...\} \subset W_2 \cap h(X)$), each of them is nonempty. We get a contradiction, because $h(X)$ is connected.

Furtheremore, prove that every connected saubspace C of $h(X)$ containing two distinct points $\xi', \xi'' \in S_k^-$, contains also any point of the open arc in S_k^- with end-points ξ' and ξ''. Indeed, suppose there is a point ξ''' of the open arc in S_k^- with end-points ξ' and ξ'' such that $\xi''' \notin C$. It is clear, that $h(X) \setminus \{\xi_0, \xi'''\}$ can be represented as the union of two open in X subsets W_1 and W_2 with $\xi' \subset W_1$, $\xi'' \subset W_2$, $W_1 \cap W_2 = \varnothing$ and $W_1 \cup W_2 = h(X) \setminus \{\xi_0, \xi'''\}$. Evidently, $W_1 \cap C$ and $W_2 \cap C$ are disjoint, open subsets of C and $C = (W_1 \cap C) \cup (W_2 \cap C)$. Besides (as $\xi' \in W_1 \cap C$ and $\xi'' \in W_2 \cap C$), each of them is nonempty. We get a contradiction, because C is connected.

Now take any point $\psi \in S_k^-$. As $\psi \notin X_1$, then (by Proposition 4) $h^{-1}(\psi) \in X_2$. Clearly, there is (unique) $n \in \mathbb{Z}$ with $h^{-1}(\psi) \in S_n$. Since S_k^- is an open subset of $h(X)$ containing the point ψ and $h(h^{-1}(\psi)) = \psi$, then by continuity of h there exists a neighborhood V of the point $h^{-1}(\psi)$ in X such that $h(V) \subset S_k^-$. Take any point $\lambda \in V \cap S_n$ different from the point $h^{-1}(\psi)$ and let $\psi' = h(\lambda) \in S_k^-$. Obviously, $\psi' \neq \psi$. Since $h^{-1}(\psi), \lambda \in S_n$, by Proposition 1, there exist two connected subsets C_1 and C_2 of X such, that $C_1 \cap C_2 = \{h^{-1}(\psi), \lambda\}$. Hence, according to the Assertion above, we should have:
$h(C_1) \cap h(C_2) = \{h(h^{-1}(\psi)), h(\lambda)\} = \{\psi, \psi'\}$.

On the other hand, let μ be any point of the open arc in S_k^- with end-points ψ and ψ'. Obviously, $\mu \neq \psi$ and $\mu \neq \psi'$. Since the connected subsets $h(C_1)$ and $h(C_2)$ of $h(X)$ both contain the points ψ and ψ', then (as it is already shown above) $h(C_1) \cap h(C_2) \supset \{\psi, \mu, \psi'\}$ which contradicts the equality $h(C_1) \cap h(C_2) = \{\psi, \psi'\}$.

Consequently, our assumption that $h(X) \neq X$ is not true. Hence, $h(X) = X$, i.e., X is topologically finite. □

Example. Below a topologically infinite locally compact and path connected subspace Y of the real plane, which is the union of its two topologically finite locally compact and path connected subspaces Y_1 and Y_2, is constructed.

For any $n \in \mathbb{Z}_{even}$ denote:

$$S_n^1 = \left\{(x, y) \in \mathbb{R}^2 : \left\| \left(n, \frac{1}{2}\right) - (x, y) \right\| = \frac{1}{2}\right\} \text{ and } S_n^2 = \left\{(x, y) \in \mathbb{R}^2 : \left\| \left(n, \frac{5}{2}\right) - (x, y) \right\| = \frac{1}{2}\right\}.$$

For any $n \in \mathbb{Z}_{odd}$ denote:

$$R_n = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; n - \frac{1}{2} \le x \le n + \frac{1}{2};\ y = 0\right\} \cup \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x = n - \frac{1}{2};\ 0 \le y \le 3\right\} \cup$$

$$\cup \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; n - \frac{1}{2} \le x \le n + \frac{1}{2};\ y = 3\right\} \cup \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x = n + \frac{1}{2};\ 0 \le y \le 3\right\}.$$

Thus S_n^1 (respectively, S_n^2) is the circumference of the circle with the center at the point $\left(n, \frac{1}{2}\right)$ (respectively, $\left(n,\frac{5}{2}\right)$) and radius $\frac{1}{2}$. As to R_n, it is the boundary of the rectangle with vertexis $\left(n-\frac{1}{2},0\right)$, $\left(n-\frac{1}{2},3\right)$, $\left(n+\frac{1}{2},3\right)$ and $\left(n+\frac{1}{2},0\right)$.

Let, $Y=\left(\bigcup_{n\in\mathbb{Z}_{even}} S_n^1\right)\cup\left(\bigcup_{n\in\mathbb{Z}_{even}} S_n^2\right)\cup\left(\bigcup_{n\in\mathbb{Z}_{odd}} R_n\right)$; $Y_1=\left(\bigcup_{n\in\mathbb{Z}_{even}} S_n^1\right)\cup\left(\bigcup_{n\in\mathbb{Z}_{odd}} R_n\right)$ and $Y_2=\left(\bigcup_{n\in\mathbb{Z}_{even}} S_n^2\right)\cup\left(\bigcup_{n\in\mathbb{Z}_{odd}} R_n\right)$.

Clearly, Y, Y_1 and Y_2 are locally compact and path connected subspaces of $\mathbb{R}^2$.
One can easily see that both Y_1 and Y_2 are homeomorphic to the space X. Hence, by Theorem 1, Y_1 and Y_2 are topologically finite. Besides, $Y=Y_1\cup Y_2$.
However, the space Y is topologically infinite. Indeed, denote: $S_0^{1+}=\left\{(x,y)\in S_0^1 \ : \ y>\frac{1}{2}\right\}$ and $S_0^{2-}=\left\{(x,y)\in S_0^2 \ : \ y<\frac{5}{2}\right\}$.
It is not difficult to check that the space Y is homeomorphic to the following its proper subspace:

$$Y\setminus\left(S_0^{1+}\cup S_0^{2-}\cup\left\{(x,y)\in\mathbb{R}^2 : x=-\frac{1}{2};\ \frac{1}{2}<y<\frac{5}{2}\right\}\cup\left\{(x,y)\in\mathbb{R}^2 : x=\frac{1}{2};\ \frac{1}{2}<y<\frac{5}{2}\right\}\right). \ \square$$

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ტოპოლოგიურად სასრული სივრცეების შესახებ

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საქართველოს საპატრიარქოს წმიდა ანდრია პირველწოდებულის ქართული უნივერსიტეტი, თბილისი

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